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Mixed derivatives and the convergence of the difference methods

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§ **1.** In this paper we shall deal with the equation

$$\frac{\partial u}{\partial t} = f\left(t, x, u, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \frac{\partial^2 u}{\partial x_2^2}\right). \tag{1.1}$$

where $x = (x_1, x_2)$.

There are serious complications in the proofs of convergence of the difference methods when the mixed derivative $\frac{\partial^2 u}{\partial x_1 \partial x_2}$ is replaced by the "large" difference quotient

$$u^{M12} = \frac{1}{4h^2}\left(u^{12(M)} - u^{-12(M)} - u^{1-2(M)} + u^{-1-2(M)}\right), \tag{1.2}$$

in the corresponding difference equation, cf. Fig.1.

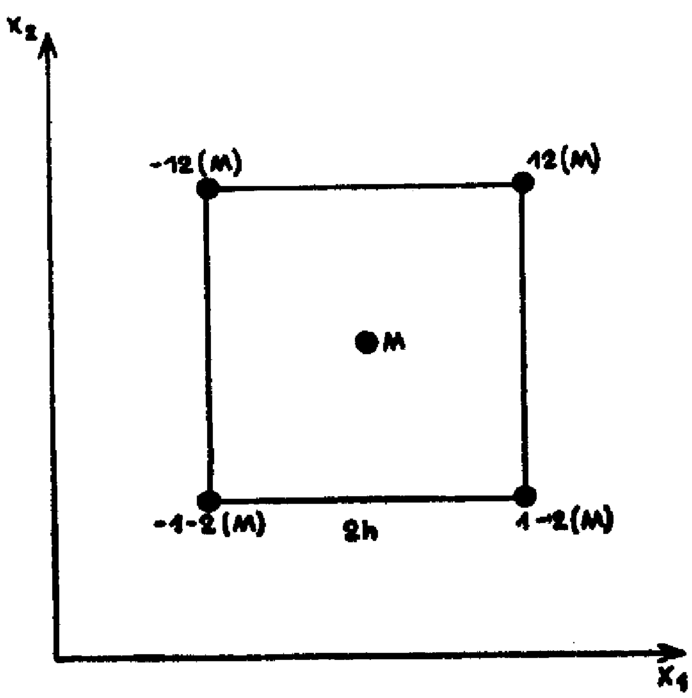

Fig. 1. The nodal points with indices $12(M)$, $-12(M)$, 1—$2(M)$, -1—$2(M)$ and M

The main idea of the proof of convergence can be explained in a short form when the equation

$$(1.3)\qquad \frac{\partial u}{\partial t} = f\left(t, x, u, \frac{\partial^2 u}{\partial x_1 \partial x_2}\right),$$

is regarded.

We shall prove that under the assumption

$$(1.4)\qquad 0 < g_{12} \leqslant \frac{\partial f}{\partial q_{12}} \leqslant \mathcal{G}_{12},$$

the difference method for the equation (1.3) is convergent. The error estimate will be given at the end the proof of Theorem 1.

The expression

$$(1.5)\qquad \tfrac{1}{2}(v^{-12(M)} + v^{1-2(M)}),$$

represents the arithmetic mean of the values $v^{-12(M)}$ and $v^{1-2(M)}$, which enter into the mixed difference quotient of the second order with the negative signs.

We shall introduce the arithmetic mean (1.5) on the left-hand side of the difference equation (2.5) and in the formula (3.4). These two places will turn out to be decisive in establishing the convergence of the method.

The equation (1.1) will be investigated in the next paper as the special case of the equation with p independent space variables x_j $(j = 1, 2, ..., p)$.

§ **2.** We shall suppose that the function $f(t, x, u, q_{12})$, $x = (x_1, x_2)$, is of the class C^1 in the set $\mathcal{D}_1$: $0 \leqslant t \leqslant T$, $0 \leqslant x_j \leqslant \alpha$, $-\infty < u < +\infty$, $-\infty < q_{12} < +\infty$ $(j = 1, 2)$.

We consider the following boundary problem in the set $\mathcal{D}$: $0 \leqslant t \leqslant T$, $0 \leqslant x_j \leqslant \alpha$ $(j = 1, 2)$:

$$(2.1)\qquad \frac{\partial u}{\partial t} = f\left(t, x, u, \frac{\partial^2 u}{\partial x_1 \partial x_2}\right),$$

$$(2.2)\qquad \begin{cases} u(0, x) = \varphi_0(x) & \\ u(t, x) = \varphi_j(t, x), & \text{for } x_j = 0 \\ u(t, x) = \psi_j(t, x), & \text{for } x_j = \alpha, \\ (j = 1, 2). & \end{cases}$$

We shall suppose that the solution $u(t, x)$ of the problem (2.1), (2.2) exists and is of the class C^2 in the set $\mathcal{D}$.

We assume also that

$$(2.3)\qquad \left|\frac{\partial f}{\partial u}\right| \leqslant \mathcal{L},$$

$$(2.4)\qquad 0 < q_{12} \leqslant \frac{\partial f}{\partial q_{12}} \leqslant \mathcal{G}_{12},$$

in the set $\mathcal{D}_1$.

The corresponding difference equation is of the explicit type and will be written in the form

$$\frac{1}{k}\cdot\left[v^{\omega(M)}-\frac{1}{2}\left(v^{-12(M)}+v^{1-2(M)}\right)\right]=f(t^{\mu},x^{m},v^{M},v^{M12}),\tag{2.5}$$

cf. Fig. 2.

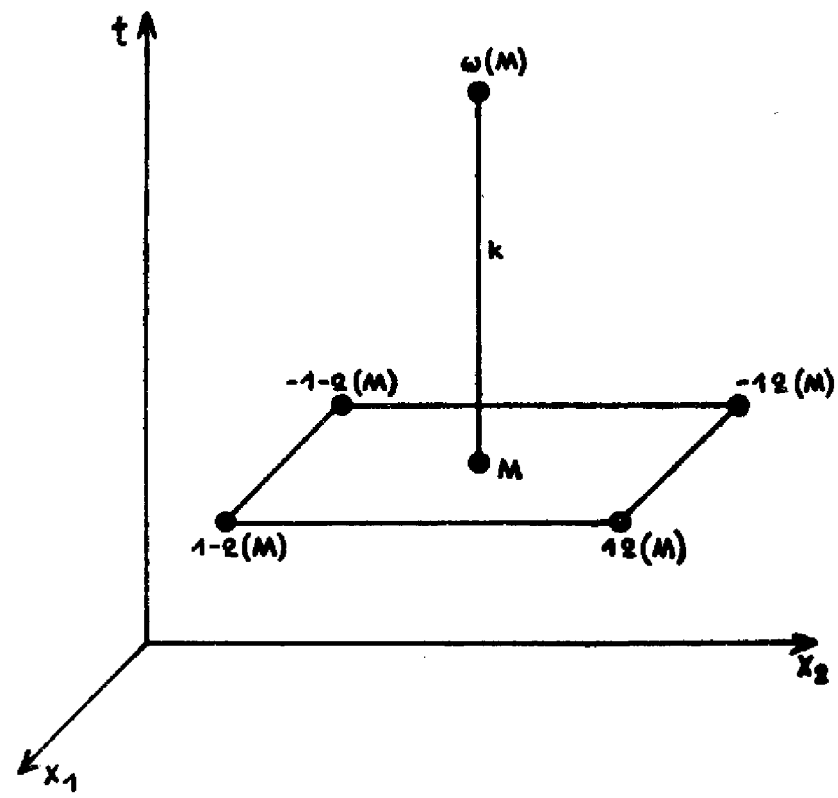

Fig. 2. The nodal points with indices $\omega(M)$, M, $12(M)$, $-12(M)$, 1—$2(M)$ and -1—$2(M)$

Here we use the notation $M=(\mu,m)$ of the paper [1], and v^{M12} denotes the "large" difference quotient of the second order:

$$v^{M12}=\frac{1}{4h^2}\left(v^{12(M)}-v^{-12(M)}-v^{1-2(M)}+v^{-1-2(M)}\right).\tag{2.6}$$

The boundary conditions are induced by the boundary conditions (2.2) and have the form

$$\begin{cases} v^M=\varphi_0(x^m), & \text{for } M=(0,m),\\ v^M=\varphi_j(x^m), & \text{for } m_j=0,\\ v^M=\psi_j(x^m), & \text{for } m_j=N,\\ (j=1,2), & \end{cases}\tag{2.7}$$

where $hN=\alpha$.

The mesh size h for the space coordinates x_j $(j=1,2)$ and k for the time coordinate t satisfy the condition

$$\frac{1}{2k}-\frac{1}{4h^2}\cdot\mathcal{G}_{12}\geqslant 0,\tag{2.8}$$

or

$$k\leqslant\frac{2}{\mathcal{G}_{12}}\cdot h^2.\tag{2.9}$$

We define the error η^M by

$$(2.10)\qquad \frac{1}{k}\cdot\left[u^{\omega(M)}-\frac{1}{2}\left(u^{-12(M)}+u^{1-2(M)}\right)\right]=f(t^\mu, x^m, u^M, u^{M12})+\eta^M,$$

and we have

$$(2.11)\qquad \varepsilon(h,k)\to 0,\ \text{as}\ h,k\to 0,$$

where

$$(2.12)\qquad \varepsilon(h,k)=\max_M|\eta^M|.$$

(2.11) means that the difference equation (2.5) is consistent with the differential equation (2.1).

We define also the error

$$(2.13)\qquad r^M=u^M-v^M.$$

§ 3. THEOREM 1. *Under the assumptions of* § 2 *the difference method is convergent.*

Proof. Let us introduce the maximal values

$$(3.1)\qquad s^\mu=\max_m r^{\mu,m}=r^{\mu,b}=r^B,$$

$$(3.2)\qquad s^{\mu+1}=\max_m r^{\mu+1,m}=r^{\mu+1,a}=r^{\omega(A)}.$$

Then we can write

$$(3.3)\qquad s^{\mu\sim}=\frac{1}{k}(s^{\mu+1}-s^\mu)=\frac{1}{k}(r^{\omega(A)}-r^B),$$

or

$$(3.4)\qquad s^{\mu}=\frac{1}{k}\left[r^{\omega(A)}-\frac{1}{2}\left(r^{-12(A)}+r^{1-2(A)}\right)\right]+\frac{1}{k}\left[\frac{1}{2}\left(r^{-12(A)}+r^{1-2(A)}\right)-r^B\right].$$

The first square bracket in the formula (3.4) can be calculated with the aid of the equations (2.10) and (2.5). To this end we subtract the equations (2.10) and (2.5), we apply the mean value theorem and we get

$$(3.5)\qquad s^\mu=\eta^A+\frac{\partial f}{\partial u}(\sim)\cdot r^A+\frac{\partial f}{\partial q_{12}}(\sim)\cdot\frac{1}{4h^2}\cdot\left(r^{12(A)}-r^{-12(A)}-r^{1-2(A)}+r^{-1-2(A)}\right)+$$
$$+\frac{1}{k}\cdot\left[\frac{1}{2}\left(r^{-12(A)}+r^{1-2(A)}\right)-r^B\right],$$

the derivatives being taken at the suitable point $(\sim)$.

We can introduce now the maximal value r^B at suitable places in (3.5) and we can write

$$(3.6)\qquad s^{\mu\sim} = \eta^A + \frac{\partial f}{\partial u}(\sim)\cdot r^A + \frac{\partial f}{\partial q_{12}}(\sim)\cdot\frac{1}{4h^2}\cdot[(r^{12(A)}-r^B)-(r^{-12(A)}-r^B)+$$
$$-(r^{1-2(A)}-r^B)+(r^{-1-2(A)}-r^B)]+\frac{1}{k}\cdot\left[\frac{1}{2}(r^{-12(A)}-r^B)+\frac{1}{2}(r^{1-2(A)}-r^B)\right].$$

We collect terms in the formula (3.6) and arrive at the equation

$$(3.7)\qquad s^{\mu\sim} = \eta^A + \frac{\partial f}{\partial u}(\sim)\cdot r^A + \frac{\partial f}{\partial q_{12}}(\sim)\cdot\frac{1}{4h^2}\cdot[(r^{12(A)}-r^B)+$$
$$+(r^{-1-2(A)}-r^B)]+(r^{-12(A)}-r^B)\cdot\left(\frac{\partial f}{\partial q_{12}}(\sim)\cdot\frac{-1}{4h^2}+\frac{1}{2k}\right)+$$
$$+(r^{1-2(A)}-r^B)\cdot\left(\frac{\partial f}{\partial q_{12}}(\sim)\cdot\frac{-1}{4h^2}+\frac{1}{2k}\right).$$

There is no difficulty in (3.7) with terms corresponding to the nodal points with indices $12(A)$ and $-1-2(A)$ (these terms enter into the difference expression (2.6) with the sign $+$) (cf. Fig. 3). In fact, the second line in (3.7) is non-positive:

$$(3.8)\qquad \frac{\partial f}{\partial q_{12}}(\sim)\cdot\frac{1}{4h^2}\cdot[(r^{12(A)}-r^B)+(r^{-1-2(A)}-r^B)]\leqslant 0,$$

and can be dropped.

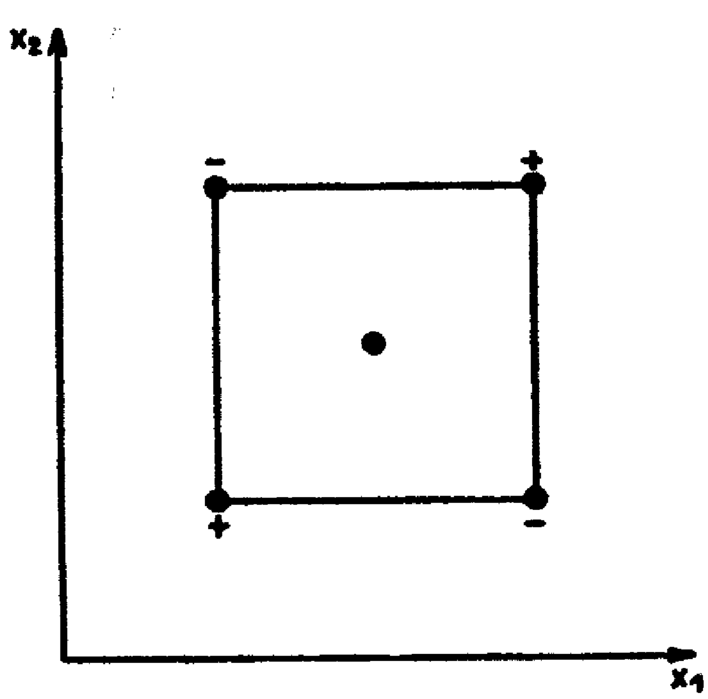

Fig. 3. The signs of the corresponding terms in the "large" difference expression for the mixed derivative $\dfrac{\partial^2 u}{\partial x_1 \partial x_2}$

The problem arises in (3.7) with terms corresponding to the nodal points with indices $-12(A)$ and $1-2(A)$ (these terms enter into the difference expression (2.6) with the sign $-$) (cf. Fig. 3).

But we have

$$\frac{\partial f}{\partial q_{12}}(\sim)\cdot\frac{-1}{4h^2}+\frac{1}{2k}\geqslant\frac{-1}{4h^2}\cdot\mathscr{G}_{12}+\frac{1}{2k}\geqslant 0, \tag{3.9}$$

because of the assumption (2.8). This means that the third and fourth line in (3.7) are non-positive

$$(r^{-12(A)}-r^B)\cdot\left(\frac{\partial f}{\partial q_{12}}(\sim)\cdot\frac{-1}{4h^2}+\frac{1}{2k}\right)\leqslant 0, \tag{3.10}$$

$$(r^{1-2(A)}-r^B)\cdot\left(\frac{\partial f}{\partial q_{12}}(\sim)\cdot\frac{-1}{4h^2}+\frac{1}{2k}\right)\leqslant 0, \tag{3.11}$$

and can be dropped.

Thus (3.7) reduces to the difference inequality

$$s^{\mu\sim}\leqslant\eta^A+\frac{\partial f}{\partial u}(\sim)\cdot r^A. \tag{3.12}$$

In a similar way we can introduce the minimum values

$$z^\mu=\min_m r^{\mu,m}=r^{\mu,d}=r^{\mathscr{D}}, \tag{3.13}$$

$$z^{\mu+1}=\min_m r^{\mu+1,m}=r^{\mu+1,c}=r^{\omega(\mathscr{C})}, \tag{3.14}$$

and we obtain the difference inequality

$$z^{\mu\sim}\geqslant\eta^{\mathscr{C}}+\frac{\partial f}{\partial u}(\sim)\cdot r^{\mathscr{C}}. \tag{3.15}$$

From (3.15) and (3.12) it follows that

$$s^{\mu\sim}\leqslant\mathscr{L}\cdot|r^A|+\varepsilon(h,k), \tag{3.16}$$

and

$$z^{\mu\sim}\geqslant-\mathscr{L}\cdot|r^{\mathscr{C}}|-\varepsilon(h,k). \tag{3.17}$$

The inequalities (3.16) and (3.17) are satisfied if the nodal point with indices A or C is on the plane $x_j=0$ or $x_j=\alpha$ $(j=1,2)$. Then we have $s^{\mu+1}=0$ and (3.16) reduces to $-\frac{1}{k}s^\mu\leqslant\mathscr{L}\cdot|r^A|+\varepsilon(h,k)(s^\mu\geqslant 0)$. In a similar way (3.17) becomes $-\frac{1}{k}z^\mu\geqslant-\mathscr{L}\cdot|r^{\mathscr{C}}|-$ $-\varepsilon(h,k)(z^\mu\leqslant 0)$.

We proceed now as in the paper [2], cf. [2] Lemma 3 and Lemma 4.

We define first

$$R^\mu=\max_m|r^M|,\quad\text{for } M=(\mu,m), \tag{3.18}$$

and we obtain

(3.19) $$R^{\mu\sim} \leqslant \max(s^{\mu\sim}, -z^{\mu\sim}),$$

because of Lemma 3, and

(3.20) $$R^{\mu\sim} \leqslant \mathscr{L} \cdot R^{\mu} + \varepsilon(h, k), R^0 = 0,$$

because of Lemma 4. This yields

(3.21) $$|r^M| \leqslant \frac{\varepsilon(h, k)}{\mathscr{L}} \cdot (l^{\mathscr{L}k\mu} - 1),$$

for $M = (\mu, m)$ and $\mu = 0, 1, \ldots, N_1$, where $kN_1 = \alpha$.

The convergence of the difference method follows from the error estimate (3.21) and the condition (2.11).

This completes the proof of Theorem 1.

References

[1] Z. Kowalski, *A difference method for a non-linear parabolic differential equation without mixed derivatives*, Ann. Polon. Math., 20 (1968), 167—177.

[2] Z. Kowalski, *On the difference method for a non-linear system of parabolic differential equations without mixed derivatives*, Bull. Acad. Polon. Sc.: Sér. Math. Astr. Phys., 16 (1968), 303—310.

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