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On the mixed derivatives and the convergence of the difference methods

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§ **1.** In this paper we shall consider the differential equation

$$\frac{\partial u}{\partial t} = f\left(t, x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right), \tag{1.1}$$

where $x = (x_1, x_2, \ldots, x_p)$, $\frac{\partial u}{\partial x}$ denotes the vector of derivatives of the first order and $\frac{\partial^2 u}{\partial x^2}$ is the $p \times p$ matrix $\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)$ $(i, j = 1, \ldots, p)$ of the derivatives of the second order.

We prove that under suitable assumptions the difference method is convergent. The error estimate is given at the end of the proof of Theorem 1.

It should be pointed out that the convergence of the method is obtained regardless of whether the matrix $\left(\frac{\partial f}{\partial q_{ij}}\right)$ $(i, j = 1, \ldots, p)$ possesses the dominating diagonal line or not. Also there is no need to form the "small" difference expressions of the second order depending on the sign of the derivatives $\frac{\partial f}{\partial q_{ij}}$, since we shall use the "large" difference expressions of the second order of the form

$$v^{Mij} = \frac{1}{4h^2} \cdot (v^{ij(M)} - v^{-ij(M)} - v^{i-j(M)} + v^{-i-j(M)}), \tag{1.2}$$

and the symmetric difference quotients

$$v^{Mjj} = \frac{1}{h^2}(v^{j(M)} - 2v^M + v^{-j(M)}), \tag{1.3}$$

cf. Fig. 1.

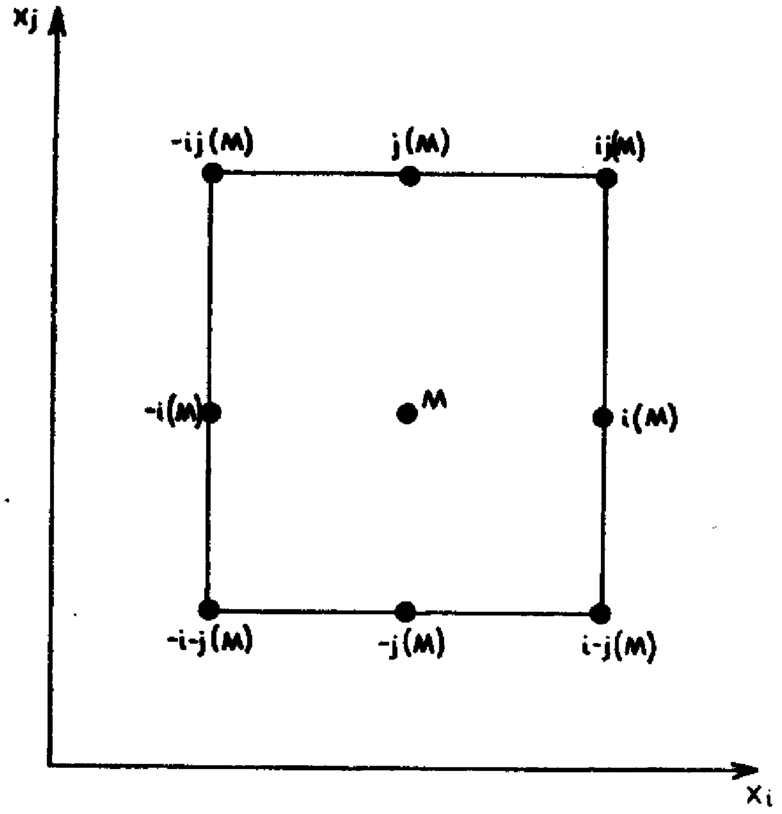

Fig. 1. The nodal points with indices M, $ij(M), \dots, -i-j(M)$

A word concerning the difficulties inherent to the problem of convergence is in order. The expression

$$(1.4)\qquad \frac{1}{p^2-p}\cdot\sum_{i=1}^{p}\sum_{\substack{j=1\\ i\neq j}}^{p}\frac{1}{3}\left(v^{-ij(M)}+v^{M}+v^{i-j(M)}\right),$$

on the left-hand side of the difference equation (2.5) and in the formula (3.4), represents the arithmetic mean of the values

$$(1.5)\qquad v^{-ij(M)},\ v^{M},\ v^{i-j(M)}\quad (i,j=1,\dots,p)(i\neq j),$$

which enter into the diffference expressions of the second order (1.2) and (1.3) with the negative signs, cf. Fig. 2.

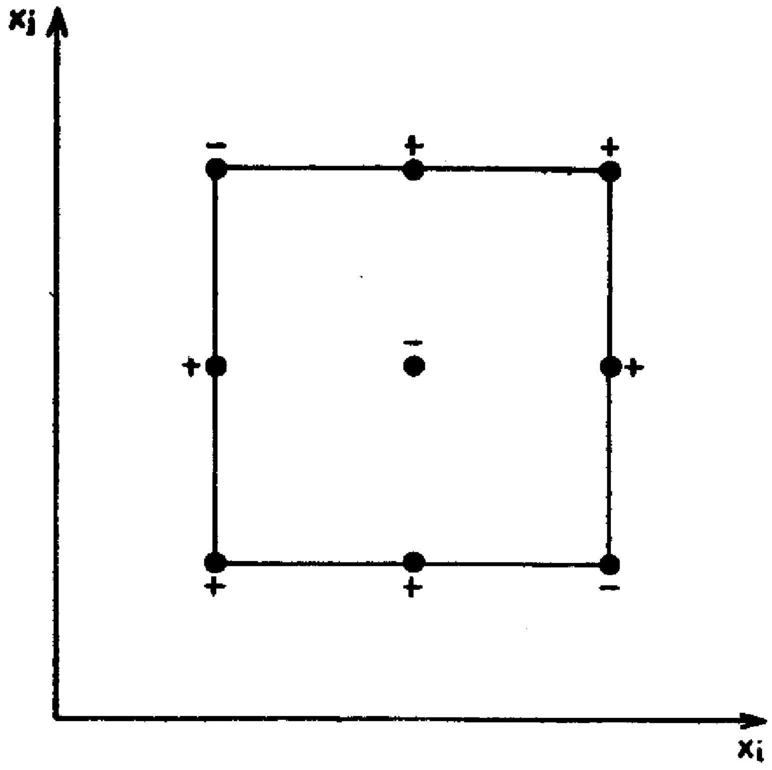

Fig. 2. The signs of the corresponding terms in the difference expressions of the second order

An inspection of (3.14) suffices to show that the negative coefficients of the values (1.5) are decisive in establishing the convergence of the difference method.

§ 2. We shall assume that the function $f(t, x, u, \overset{1}{q}, \overset{2}{q})$

$$x = (x_1, \dots, x_p), \overset{1}{q} = (\overset{1}{q}_1, \dots, \overset{1}{q}_p), \overset{2}{q} = (q_{ij}) \quad (i, j = 1, \dots, p),$$

is of the class C^1 in the set $\mathscr{D}_1$: $0 \leqslant t \leqslant T$, $0 \leqslant x_j \leqslant \alpha$, $-\infty < u < +\infty$, $-\infty < \overset{1}{q}_j < +\infty$, $-\infty < q_{ij} < +\infty$ $(i, j = 1, \dots, p)$.

We consider the following boundary problem in the set

$$\mathscr{D}\colon\ 0 \leqslant t \leqslant T,\ 0 \leqslant x_j \leqslant \alpha \quad (j = 1, 2, \dots, p):$$

$$\frac{\partial u}{\partial t} = f\left(t, x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right), \tag{2.1}$$

$$\begin{cases} u(0, x) = \varphi_0(x), \\ u(t, x) = \varphi_j(t, x), & \text{for } x_j = 0, \\ u(t, x) = \psi_j(t, x), & \text{for } x_j = \alpha, \\ (j = 1, 2, \dots, p). \end{cases} \tag{2.2}$$

In the equation (2.1) $\dfrac{\partial u}{\partial x}$ denotes the vector $\dfrac{\partial u}{\partial x} = \left(\dfrac{\partial u}{\partial x_1}, \dfrac{\partial u}{\partial x_2}, \dots, \dfrac{\partial u}{\partial x_p}\right)$, and $\dfrac{\partial^2 u}{\partial x^2}$ denotes the $p \times p$ matrix of the partial derivatives $\dfrac{\partial^2 u}{\partial x^2} = \left(\dfrac{\partial^2 u}{\partial x_i \partial x_j}\right)$ $(i, j = 1, 2, \dots, p)$.

We shall assume that the solution $u(t, x)$ of the problem (2.1), (2.2) exists and is of the class C^2 in the set $\mathscr{D}$.

We assume also that

$$\left|\frac{\partial f}{\partial u}\right| \leqslant \mathscr{L}, \quad \left|\frac{\partial f}{\partial \overset{1}{q}_j}\right| \leqslant \Gamma_j \quad (j = 1, \dots, p) \tag{2.3}$$

$$0 < g_{ij} \leqslant \frac{\partial f}{\partial q_{ij}} \leqslant \mathscr{G}_{ji} \quad (i, j = 1, \dots, p), \tag{2.4}$$

in the set $\mathscr{D}_1$.

The corresponding difference equation is of the explicite type and will be written in the form

$$\frac{1}{k} \cdot \left[v^{\omega(M)} - \frac{1}{p^2 - p} \cdot \sum_{i=1}^{p} \sum_{\substack{j=1 \\ i \neq j}}^{p} \frac{1}{3} \left(v^{-ij(M)} + v^M + v^{i-j(M)}\right)\right] = \tag{2.5}$$

$$= f(t^\mu, x^m, v^M, v^{M\overline{1}}, v^{M\overline{2}}),$$

cf. Fig. 3.

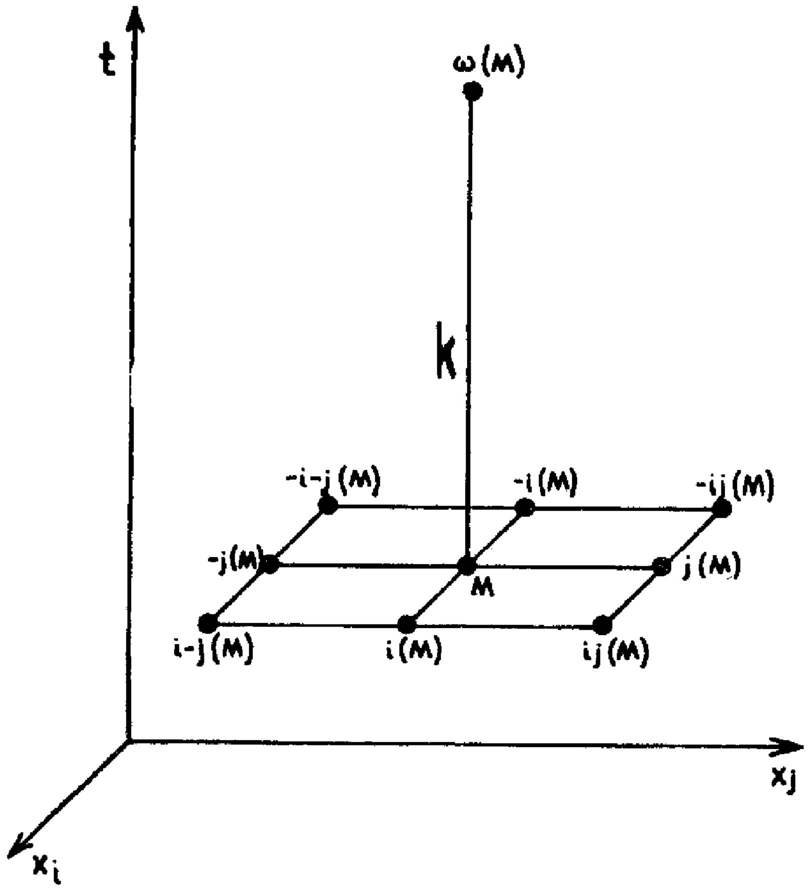

Fig. 3. The nodal points with indices M, $ij(M), \dots, -i-j(M)$, $\omega(M)$

Here we use the notation $M = (\mu, m)$ of the paper [1], $v^{M\bar{1}}$ denotes the vector of the symmetric difference quotients of the first order

$$v^{M\bar{1}} = (v^{M1}, v^{M2}, \dots, v^{Mp}), \tag{2.6}$$

where

$$v^{Mj} = \frac{1}{2h}\cdot(v^{j(M)} - v^{-j(M)}) \qquad (j = 1, 2, \dots, p), \tag{2.7}$$

and $v^{M\bar{2}}$ the $p \times p$ matrix $v^{M\bar{2}} = (v^{Mij})$ $(i, j = 1, \dots, p)$, whose elements are

$$v^{Mjj} = \frac{1}{h^2}(v^{j(M)} - 2v^M + v^{-j(M)}) \qquad (j = 1, 2, \dots, p), \tag{2.8}$$

and

$$\begin{cases} v^{Mij} = \dfrac{1}{4h^2}(v^{ij(M)} - v^{-ij(M)} - v^{i-j(M)} + v^{-i-j(M)}), \\ (i \neq j) \ (i, j = 1, 2, \dots, p), \end{cases} \tag{2.9}$$

respectively.

The boundary conditions are induced by the boundary conditions (2.2) and have the form

$$\begin{cases} v^M = \varphi_0(x^m), & \text{for } M = (\mu, m), \\ v^M = \varphi_j(t^\mu, x^m), & \text{for } m_j = 0, \\ v^M = \psi_j(t^\mu, x^m), & \text{for } m_j = N, \\ (j = 1, 2, \dots, p), \end{cases} \tag{2.10}$$

where $hN = \alpha$.

The mesh size h for the space coordinates x_j ($j = 1, \dots, p$) and k for the time coordinate t satisfy the conditions

$$g_{jj} \cdot \frac{1}{h} - \Gamma_j \cdot \frac{1}{2} \geqslant 0 \quad (j = 1, 2, \dots, p), \tag{2.11}$$

and

$$\frac{1}{3k} \cdot \frac{1}{p^2 - p} - \frac{1}{4h^2} \cdot \mathscr{G}_{ij} \geqslant 0 \quad (i \neq j), \tag{2.12}$$

$$\frac{1}{3k} - \frac{2}{h^2} \cdot \sum_{j=1}^{p} \mathscr{G}_{jj} \geqslant 0, \tag{2.13}$$

for $i, j = 1, 2, \dots, p$.

This means that

$$k \leqslant 4h^2 \cdot \frac{1}{p^2 - p} \cdot \frac{1}{3} \cdot \frac{1}{\mathscr{G}_{ij}} \quad (i \neq j), \tag{2.15}$$

and

$$k \leqslant h^2 \cdot \frac{1}{3} \cdot \frac{1}{2 \cdot \sum\limits_{j=1}^{p} \mathscr{G}_{jj}}, \tag{2.16}$$

for $i, j = 1, 2, \dots, p$.

We define the error η^M by

$$\frac{1}{k} \cdot \left[u^{\omega(M)} - \frac{1}{p^2 - p} \cdot \sum_{i=1}^{p} \sum_{\substack{j=1 \\ i \neq j}}^{p} \frac{1}{3} \left(u^{-ij(M)} + u^M + u^{i-j(M)} \right) \right] = \tag{2.17}$$

$$= f(t^\mu, x^m, u^M, u^{M\bar{1}}, u^{M\bar{2}}) + \eta^M,$$

and we have

$$\varepsilon(h, k) \to 0, \quad \text{as } h, k \to 0, \tag{2.18}$$

where

$$\varepsilon(h, k) = \max_M |\eta^M|. \tag{2.19}$$

(2.18) means that the difference equation (2.5) is consistent with the differential equation (2.1).

We define also the error

$$r^M = u^M - v^M. \tag{2.20}$$

§ 3. THEOREM 1. *Under the assumptions of* § 2 *the difference method is convergent.*

Proof. We introduce the maximal values

$$s^{\mu} = \max_{m} r^{\mu,m} = r^{\mu,b} = r^{B}, \tag{3.1}$$

$$s^{\mu+1} = \max_{m} r^{\mu+1,m} = r^{\mu+1,a} = r^{\omega(A)}, \tag{3.2}$$

and we can write

$$s^{\mu\sim} = \frac{1}{k}(s^{\mu+1} - s^{\mu}) = \frac{1}{k}(r^{\omega(A)} - r^{B}), \tag{3.3}$$

or

$$s^{\mu\sim} = \frac{1}{k}\left[r^{(\omega)A} - \frac{1}{p^2-p}\cdot\sum_{\substack{i=1\\ i\neq j}}^{p}\sum_{j=1}^{p}\frac{1}{3}(r^{-ij(A)}+r^{A}+r^{i-j(A)})\right]+ \tag{3.4}$$

$$+\frac{1}{k}\left[\frac{1}{p^2-p}\sum_{\substack{i=1\\ i\neq j}}^{p}\sum_{j=1}^{p}\frac{1}{3}(r^{-ij(A)}+r^{A}+r^{i-j(A)})-r^{B}\right].$$

The first square bracket in the formula (3.4) can be calculated with the aid of the equation (2.17) and (2.5). To this end we subtract the equations (2.17) and (2.5), we apply the mean value theorem and we get

$$s^{\mu\sim} = \eta^{A} + \frac{\partial f}{\partial u}(\sim)\cdot r^{A} + \sum_{j=1}^{p}\frac{\partial f}{\overset{1}{\partial} q_j}(\sim)\cdot\frac{1}{2h}\cdot(r^{j(A)} - r^{-j(A)})+ \tag{3.5}$$

$$+\sum_{j=1}^{p}\frac{\partial f}{\partial q_{jj}}(\sim)\cdot\frac{1}{h^2}(r^{j(A)} - 2r^{A} + r^{-j(A)})+$$

$$+\sum_{\substack{i=1\\ i\neq j}}^{p}\sum_{j=1}^{p}\frac{\partial f}{\partial q_{ij}}(\sim)\cdot\frac{1}{4h^2}(r^{ij(A)} - r^{-ij(A)} - r^{i-j(A)} + r^{-i-j(A)})+$$

$$+\frac{1}{k}\left[\frac{1}{p^2-p}\sum_{\substack{i=1\\ i\neq j}}^{p}\sum_{j=1}^{p}\frac{1}{3}(r^{-ij(A)}+r^{A}+r^{i-j(A)})-r^{B}\right].$$

We can now introduce r^{B} at suitable places and we obtain

$$s^{\mu\sim} = \eta^{A} + \frac{\partial f}{\partial u}(\sim)\cdot r^{A} + \mathscr{C}_1 + \mathscr{C}_2, \tag{3.6}$$

where

$$
\mathscr{C}_1 = \sum_{\substack{i=1 \\ i \neq j}}^{p} \sum_{j=1}^{p} \frac{\partial f}{\partial q_{ij}}(\sim) \cdot \frac{1}{4h^2} \left[(r^{ij(A)} - r^B) + (r^{-i-j(A)} - r^B) \right] + \tag{3.7}
$$

$$
+ \sum_{j=1}^{p} \frac{\partial f}{\partial q_{jj}}(\sim) \cdot \frac{1}{h^2} \left[(r^{j(A)} - r^B) + (r^{-j(A)} - r^B) \right] +
$$

$$
+ \sum_{j=1}^{p} \frac{\partial f}{\overset{1}{\partial} q_j}(\sim) \cdot \frac{1}{2h} \left[(r^{j(A)} - r^B) - (r^{-j(A)} - r^B) \right],
$$

and

$$
\mathscr{C}_2 = \sum_{\substack{i=1 \\ i \neq j}}^{p} \sum_{j=1}^{p} \frac{\partial f}{\partial q_{ij}}(\sim) \cdot \frac{1}{4h^2} \left[-(r^{-ij(A)} - r^B) - (r^{i-j(A)} - r^B) \right] + \tag{3.8}
$$

$$
+ \sum_{j=1}^{p} \frac{\partial f}{\partial q_{jj}}(\sim) \cdot \frac{1}{h^2} \cdot \left[-2(r^A - r^B) \right] +
$$

$$
+ \frac{1}{k} \cdot \frac{1}{p^2 - p} \cdot \sum_{\substack{i=1 \\ i \neq j}}^{p} \sum_{j=1}^{p} \frac{1}{3} \left[(r^{-ij(A)} - r^B) + (r^A - r^B) + (r^{i-j(A)} - r^B) \right],
$$

since in the last line of the formula (3.5) we can substitute

$$
r^B = \frac{1}{p^2 - p} \cdot \sum_{\substack{i=1 \\ i \neq j}}^{p} \sum_{j=1}^{p} r^B, \tag{3.9}
$$

and write

$$
\frac{1}{k} \cdot \left[\frac{1}{p^2 - p} \sum_{\substack{i=1 \\ i \neq j}}^{p} \sum_{j=1}^{p} \frac{1}{3} (r^{-ij(A)} + r^A + r^{i-j(A)}) - r^B \right] = \tag{3.10}
$$

$$
= \frac{1}{k} \cdot \frac{1}{p^2 - p} \cdot \sum_{\substack{i=1 \\ i \neq j}}^{p} \sum_{j=1}^{p} \frac{1}{3} \left[(r^{-ij(A)} - r^B) + (r^A - r^B) + (r^{i-j(A)} - r^B) \right].
$$

There is no difficulty in (3.7) with terms corresponding to the nodal points with indices $ij(A)$, $-i-j(A)$, $j(A)$, $-j(A)$ (these terms enter into the difference expressions with the sign +) cf. Fig. 2. In fact, the first line in (3.7) is non-positive and can be dropped. The second and third line in (3.7) can be rewritten as follows

(3.11)
$$\sum_{j=1}^{p}\frac{\partial f}{\partial q_{jj}}(\sim)\cdot\frac{1}{h^2}\cdot\left[\left(r^{j(A)}-r^{B}\right)+\left(r^{-j(A)}-r^{B}\right)\right]+$$
$$+\sum_{j=1}^{p}\frac{\partial f}{\partial \overset{1}{q}_{j}}(\sim)\cdot\frac{1}{2h}\left[\left(r^{j(A)}-r^{B}\right)-\left(r^{-j(A)}-r^{B}\right)\right]=$$
$$=\sum_{j=1}^{p}\left(r^{j(A)}-r^{B}\right)\cdot\left(\frac{\partial f}{\partial q_{jj}}(\sim)\cdot\frac{1}{h^2}+\frac{\partial f}{\partial \overset{1}{q}_{j}}(\sim)\cdot\frac{1}{2h}\right)+$$
$$+\left(r^{-j(A)}-r^{B}\right)\cdot\left(\frac{\partial f}{\partial q_{jj}}(\sim)\cdot\frac{1}{h^2}-\frac{\partial f}{\partial \overset{1}{q}_{j}}(\sim)\cdot\frac{1}{2h}\right).$$

But we have

(3.12)
$$\frac{\partial f}{\partial q_{jj}}(\sim)\cdot\frac{1}{h^2}+\frac{\partial f}{\partial \overset{1}{q}_{j}}(\sim)\cdot\frac{1}{2h}\geqslant\frac{1}{h}\left(g_{jj}\cdot\frac{1}{h}-\Gamma_j\cdot\frac{1}{2}\right)\geqslant 0,$$

(3.13)
$$\frac{\partial f}{\partial q_{jj}}(\sim)\cdot\frac{1}{h^2}-\frac{\partial f}{\partial \overset{1}{q}_{j}}(\sim)\cdot\frac{1}{2h}\geqslant\frac{1}{h}\left(g_{jj}\cdot\frac{1}{h}-\Gamma_j\cdot\frac{1}{2}\right)\geqslant 0,$$

because of the assumption (2.11), hence the right-hand side of the formula (3.11) is non-positive and can be dropped also.

The problem arises in (3.8) with terms corresponding to the nodal points with indices $-ij(A)$, A, $i-j(A)$ (these terms enter into the difference expressions with the sign $-$), cf. Fig. 2.

But we have

(3.14)
$$\mathscr{C}_2=\sum_{\substack{i=1\\ i\neq 1}}^{p}\sum_{j=1}^{p}\left(r^{-ij(A)}-r^{B}\right)\cdot\left(\frac{\partial f}{\partial q_{ij}}(\sim)\cdot\frac{-1}{4h^2}+\frac{1}{k}\cdot\frac{1}{p^2-p}\cdot\frac{1}{3}\right)+$$
$$+\sum_{\substack{i=1\\ i\neq j}}^{p}\sum_{j=1}^{p}\left(r^{i-j(A)}-r^{B}\right)\cdot\left(\frac{\partial f}{\partial q_{ij}}(\sim)\cdot\frac{-1}{4h^2}+\frac{1}{k}\cdot\frac{1}{p^2-p}\cdot\frac{1}{3}\right)+$$
$$+\left(r^{A}-r^{B}\right)\cdot\left(\sum_{j=1}^{p}\frac{\partial f}{\partial q_{jj}}(\sim)\cdot\frac{-2}{h^2}+\frac{1}{k}\cdot\frac{1}{p^2-p}\cdot\sum_{\substack{i=1\\ i\neq j}}^{p}\sum_{j=1}^{p}\frac{1}{3}\right).$$

In the first and second line of the formula (3.14) we have

(3.15)
$$\frac{\partial f}{\partial q_{ij}}(\sim)\cdot\frac{-1}{4h^2}+\frac{1}{k}\cdot\frac{1}{p^2-p}\cdot\frac{1}{3}\geqslant\frac{-1}{4h^2}\cdot\mathcal{G}_{ij}+\frac{1}{k}\cdot\frac{1}{p^2-p}\cdot\frac{1}{3}\geqslant 0,$$

because of the assumption (2.12), which means that the first and second line in (3.14) are non-positive and can be dropped.

In the third line of the formula (3.14) we have

(3.16)
$$\sum_{j=1}^{p}\frac{\partial f}{\partial q_{jj}}(\sim)\cdot\frac{-2}{h^2}+\frac{1}{k}\cdot\frac{1}{p^2-p}\cdot\sum_{i=1}^{p}\sum_{\substack{j=1\\ i\neq j}}^{p}\frac{1}{3}\geqslant\frac{-2}{h^2}\cdot\sum_{j=1}^{p}\mathcal{G}_{jj}+\frac{1}{k}\cdot\frac{1}{3}\geqslant 0,$$

because of the assumption (2.13). Hence, the third line in (3.14) is non-positive and can be dropped also.

Thus (3.7) and (3.8) reduces to

(3.17)
$$\mathcal{C}_1\leqslant 0,\qquad \mathcal{C}_2\leqslant 0,$$

and we can majorize the right-hand member in (3.6) and write

(3.18)
$$s^{\mu\sim}\leqslant\eta^A+\frac{\partial f}{\partial u}(\sim)\cdot r^A.$$

In a similar way we can introduce the minimum values

(3.19)
$$z^{\mu}=\min_m r^{\mu,m}=r^{\mu,d}=r^{\mathcal{D}},$$

(3.20)
$$z^{\mu+1}=\min_m r^{\mu+1,m}=r^{\mu+1,c}=r^{\omega(\mathcal{C})},$$

and obtain the difference inequality

(3.21)
$$z^{\mu\sim}\geqslant\eta^{\mathcal{C}}+\frac{\partial f}{\partial u}(\sim)\cdot r^{\mathcal{C}}.$$

From (3.21) and (3.18) it follows that

(3.22)
$$s^{\mu\sim}\leqslant\mathcal{L}\cdot|r^A|+\varepsilon(h,k),$$

and

(3.23)
$$z^{\mu\sim}\geqslant-\mathcal{L}\cdot|r^{\mathcal{C}}|-\varepsilon(h,k).$$

We proceed now as in the paper [2], cf. [2] Lemma 3 and Lemma 4.

We define first

(3.24)
$$R^{\mu}=\max_m|r^M|,\qquad\text{for } M=(\mu,m),$$

and obtain

(3.25)
$$R^{\mu\sim}\leqslant\max(s^{\mu\sim},-z^{\mu\sim}).$$

because of Lemma 3, and

$$R^{\mu\sim} \leqslant \mathscr{L}\cdot R^{\mu}+\varepsilon(h,k),\ R^0 = 0\,, \tag{3.26}$$

because of Lemma 4.

This yields

$$|r^M| \leqslant \frac{\varepsilon(h,k)}{\mathscr{L}}\cdot(e^{\mathscr{L}k\mu}-1)\,, \tag{3.27}$$

for $M = (\mu, m)(\mu = 0, 1, \ldots, N_1)$, where $kN_1 = T$.

The convergence of the difference method follows from the error estimate (3.27) and the condition (2.18).

This completes the proof of Theorem 1.

References

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