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The fixed point index of a translation operator of a semiflow

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Abstract. In the paper a formula for computing the fixed point index of the translation operator for a semiflow is given.

1. Introduction. The purpose of the present paper is to compute the fixed point index of the translation operator for a semiflow. The value of the index is expressed by the values of Euler characteristics of certain sets or by generalized Euler characteristics in case of BDR's (the definitions are given below).

The modern fixed point index theory was originated in 1965 by A. Dold (cf. [3]). The history of the subject may be found in [2]. In 1972 A. Granas generalized the theory to the case of non-compact ANR's. We recall that the fixed point index is an integer valued function i defined on the class

$\mathscr{C} := \{(f, U)$: U is open in a compact ANR X, f: $X \nrightarrow X$ is a (partial) continuous mapping such that $\bar{U} \subset \operatorname{dom} f$ and f has no fixed points on $\partial U\}$

and satisfying the following five axioms

(i1) (Localization) If $(f, U), (g, U) \in \mathscr{C}$ and $f(t) = g(t)$ for $t \in \bar{U}$, then $i(f, U) = i(g, U)$

(i2) (Homotopy) If f_t is a homotopy such that $(f_t, U) \in \mathscr{C}$ for each t, $0 \leqslant t \leqslant 1$, then $i(f_0, U) = i(f_1, U)$

(i3) (Additivity) If $(f, U) \in \mathscr{C}$ and U contains mutually disjoint open subsets V_j, $j = 1, 2, \ldots, k$ such that f has no fixed points in $U \setminus \bigcup_{j=1}^{k} V_j$, then $(f, V_j) \in \mathscr{C}$ for $j = 1, 2, \ldots k$ and $i(f, U) = \sum_{j=1}^{k} i(f, V_j)$

(i4) (Normalization) If X is a compact ANR and f: $X \to X$ is continuous, then $(f, X) \in \mathscr{C}$ and $i(f, X) = \Lambda(f)$, where $\Lambda(f)$ denotes the Lefschetz number of f

(i5) (Commutativity) If X, Y are compact ANR's, $f\colon X \mathrel{-\!\!\circ\!\!\!\to} Y$, $g\colon Y \mathrel{-\!\!\circ\!\!\!\to} X$ are continuous, $(gf, U) \in \mathscr{C}$, then $(fg, g^{-1}(U)) \in \mathscr{C}$ and $i(gf, U) = i(fg, g^{-1}(U))$.

The presented theory is motivated by the existence problems in the theory of dynamical systems. The obtained formula will be applied in [5] to prove the existence of stationary points and periodic trajectories for flows and differential equations.

In Part 2 we give preliminary results and definitions. In Part 3 we introduce the notion of a boundary deformation retract (BDR) and give the definition of the generalized Euler characteristic for BDR's. Main results of the paper are stated and proved in Part 4.

2. Preliminary lemmas. The following notation will be used. The set of reals is denoted by $\boldsymbol{R}$ and the set of non-negative reals by $\boldsymbol{R}^*$. (X, ϱ) denotes a fixed metric space X with the distance ϱ. For $A \subset X$, $K(A, r)$ will stand for the set $\{y \in X\colon \varrho(y, A) < r\}$. $\bar{A}$, $\operatorname{int} A$, ∂A denote the closure, the interior and the boundary of A, respectively.

To simplify the formulations we admit the empty mapping (i.e. assume that ϕ is a mapping) and define its fixed point index by

$$i(\phi, \phi) = 0 .$$

Let $f\colon \boldsymbol{R}^* \times X \mathrel{-\!\!\circ\!\!\!\to} X$ be a partial, continuous mapping and let $\operatorname{dom} f$ be its domain. For a fixed $t > 0$ define the (possibly empty) mapping (the t-translation operator of f)

$$f_t\colon \{x\colon (t, x) \in \operatorname{dom} f\} \ni x \to f_t(x) := f(t, x) \in X$$

and by $\operatorname{Fix}_t f$ the set of all fixed points of f_t

$$\operatorname{Fix}_t f := \{x \in X\colon (t, x) \in \operatorname{dom} f \text{ and } f_t(x) = x\} .$$

Definition 2.1. The mapping f will be called *the deformation on* X if it satisfies the following conditions

(2.1) $\operatorname{dom} f$ is a closed set

(2.2) $\{0\} \times X \subset \operatorname{dom} f$ and $\forall x \in X\ f(0, x) = x$

(2.3) $\forall x \in X\ I_f(x) := \{t \in \boldsymbol{R}^*\colon (t, x) \in \operatorname{dom} f\}$ is a segment in $\boldsymbol{R}^*$.

Define the function

$$\omega_f\colon X \ni x \to \sup I_f(x) \in \boldsymbol{R}^* \cup \{\infty\} .$$

LEMMA 2.1. *Assume the continuous mapping $f\colon \boldsymbol{R}^* \times X \mathrel{-\!\!\circ\!\!\!\to} X$ satisfies* (2.2) *and* (2.3). *Then dom f is closed iff the mapping ω_f is upper semi-continuous* (*u.s.c.*) *and the set $I_f(x)$ is closed for every $x \in X$.*

Proof. Assume $\operatorname{dom} f$ is closed. Then the set $I_f(x)$ is closed for every $x \in X$. To prove the upper semi-continuity of ω_f, consider the sequence $\{x_n\} \subset X$, which converges to $x \in X$. Let $t := \limsup_{n \to \infty} \omega_f(x_n)$. If $t < \infty$, then $\omega_f(x_n) < \infty$ and consequently $(\omega_f(x_n), x_n) \in \operatorname{dom} f$ for almost all n. Since $\operatorname{dom} f$ is closed, $(t, x) \in \operatorname{dom} f$ and $t \leqslant \omega_f(x)$. If $t = \infty$, then for any $s \geqslant 0$ we will find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\omega_f(x_{n_k}) \geqslant s$. From the property (2.3) it follows that $(s, x_{n_k}) \in \operatorname{dom} f$. Consequently $(s, x) \in \operatorname{dom} f$ and $\omega_f(x) = \infty$.

To prove the converse, consider a sequence $\{(t_n, x_n)\} \subset \operatorname{dom} f$, tending to (t, x). From the u.s.c. of ω_f it follows that $t \leqslant \omega_f(x)$. But $I_f(x)$ is a closed segment, so $(t, x) \in \operatorname{dom} f$. The proof is finished.

Consider the mapping

$$\Omega_f\colon \{x \in X\colon \omega_f(x) < \infty\} \ni x \to f(\omega_f(x), x) \in X$$

and denote its image by $\operatorname{Fin} f$.

Definition 2.2. The deformation f will be called *regular* if it satisfies the implication

$$f(t, x) \in \operatorname{Fin} f \Rightarrow t = \omega_f(x).$$

LEMMA 2.2. *Assume X is compact and f is a regular deformation. Then the mapping ω_f is continuous iff the set $\operatorname{Fin} f$ is closed.*

Proof. Assume ω_f is continuous. Let $\{y_n\} \subset \operatorname{Fin} f$, $y = \lim y_n$. Then $y_n = f(\omega_f(x_n), x_n)$ for some $x_n \in X$. Passing to a subsequence if necessary, we may assume that $x_n \to x$. Thus we have $y = \lim f(\omega_f(x_n), x_n) = f(\omega_f(x), x) \in \operatorname{Fin} f$.

To prove the converse, consider the sequence $x_n \to x$. Let $r_n := \omega_f(x_n)$. First suppose that at least one subsequence $\{r_{n_k}\}$ of $\{r_n\}$ tends to a finite value, and let $r := \lim r_{n_k}$. Set $y_k := f(r_{n_k}, x_{n_k})$. Then $y = \lim y_k = f(r, x) \in \operatorname{Fin} f$. Since f is regular, $r = \omega_f(x)$. Thus $r = \omega_f(x) < \infty$ and by u.s.c. of ω_f, the sequence $\{r_n\}$ is bounded. Consequently $r = \lim r_n$ and ω_f is continuous at x.

If $r_n \to \infty$ then the continuity of ω_f at x follows immediately from the upper semicontinuity of ω_f. The proof is completed.

Let Y be a subspace of X such that $\operatorname{Fin} f \subset Y$ and let g be a deformation on Y.

Definition 2.3. The mapping

$$(g \Box f)(t, x) := \begin{cases} f(t, x) & 0 \leqslant t \leqslant \omega_f(x) \\ g(t - \omega_f(x), \Omega_f(x)) & \omega_f(x) \leqslant t \leqslant \omega_f(x) + \omega_g(\Omega_f(x)) \end{cases}$$

will be called *the prolongation of f along g*.

LEMMA 2.3. *Let X be compact and let $Y \subset X$ be closed. Assume deformations f on X and g on Y are regular and satisfy the condition*

$$Y = \operatorname{Fin} f \cup \operatorname{Fin} g. \tag{2.4}$$

Then $g \Box f$ is a deformation on X.

Proof. Properties (2.2) and (2.3) obviously hold true. We will prove the property (2.1) and the continuity of $g \Box f$ simultaneously. Consider the sequence $\{(t_n, x_n)\} \subset \operatorname{dom} g \Box f$ and assume $(t_n, x_n) \to (t, x)$. If for every $N \in N$ $t_n \leqslant \omega_f(x_n)$, then (2.1) and the continuity of $g\square f$ follow at once from the definition of the prolongation. Hence it is enough to consider only the case where $t_n \geqslant \omega_f(x_n)$ for every $n \in N$.

Put $y_n := \Omega_f(x_n)$, $r_n := \omega_f(x_n)$, $s_n := t_n - \omega_f(x_n)$. Of course $s_n \leqslant \omega_g(y_n)$. Passing, if necessary, to subsequences we may assume that $r_n \to r$, $y_n \to y$, $s_n \to s$, where $s = t-r$. Since Y is closed we conclude that $(r, y) \in \operatorname{dom} f$ and $y \in Y$. From the u.s.c. of ω_g it follows that $s \leqslant \omega_g(y)$. If $y \in \operatorname{Fin} f$, then $r = \omega_f(x)$ and since f is regular, we get $y = \Omega_f(x)$. Hence we have $t = r+s \leqslant \omega_f(x) + \omega_g(\Omega_f(x))$, which means that $(t, x) \in \operatorname{dom} g\square f$. We have also

$$\lim g\big(t_n - \omega_f(x_n), \Omega_f(x_n)\big) = g(t-r, y) = g\big(t-\omega_f(x), \Omega_f(x)\big),$$

which proves the continuity of $g\square f$ at (t, x).

If $y \in \operatorname{Fin} g$, then by the regularity of g, we get $\omega_g(y) = 0$. But $s \leqslant \omega_g(y)$, hence $s = 0$. It means that $t = r \leqslant \omega_f(x)$. Consequently $(t, x) \in \operatorname{dom} g\square f$ and

$$\lim g\big(t_n - \omega_f(x_n), \Omega_f(x_n)\big) = g(0, y) = y = f(r, x) = f(t, x) = (g\square f)(t, x).$$

The proof is finished.

Definition 2.4. A regular deformation f on X will be called *a partial semiflow* on X if it satisfies the condition

(2.5) $\quad \forall x \in X\, \forall t \in I_f(x)\, \forall s \in I_f(f(t, x))\ s+t \in I_f(x)$ and $f(s, f(t, x)) = f(s+t, x)$.

It will be called simply a semiflow on X if, additionally, $\operatorname{dom} f = \boldsymbol{R}^* \times X$.

LEMMA 2.4. *Let X be compact and let $Y \subset X$ be its closed subspace. Assume f is a partial semiflow on X and g is a regular deformation on Y such that Fin g is closed. If f and g satisfy* (2.4), *then $f\square(g\square f)$ is a deformation on X.*

Proof. We proceed like in the previous lemma. Consider a sequence $\{(t_n, x_n)\} \subset \operatorname{dom} f\square(g\square f)$ and assume that $(t_n, x_n) \to (t, x)$. We may suppose that $t_n \geqslant \omega_f(x_n) + \omega_g(\Omega_f(x_n))$. Put $y_n := \Omega_f(x_n)$, $z_n := \Omega_g(y_n)$, $s_n := \omega_f(x_n)$, $r_n := \omega_g(y_n)$, $u_n := t_n - r_n - s_n$. We may assume that $y_n \to y$, $z_n \to z$, $s_n \to s$, $r_n \to r$, $u_n \to u$. First consider the case $y \in \operatorname{Fin} f$. We get then $s = \omega_f(x)$, $y = \Omega_f(x)$. Since by Lemma 2.2 and our assumptions, ω_g is continuous, we obtain $r = \omega_g(y)$. Hence $t = s+r+u \leqslant \omega_f(x) + \omega_g(y) + \omega_f(z) = \omega_{f\square(g\square f)}(x)$. Thus $(t, x) \in \operatorname{dom} f\square(g\square f)$ and

$$\begin{aligned} \lim f\square(g\square f)(t_n, x_n) &= \lim f\big(t_n - s_n - r_n, g(r_n, \Omega_f(x_n))\big) \\ &= f\big(t-s-r, g(r, \Omega_f(x))\big) = f\square(g\square f)(t, x). \end{aligned}$$

Passing to the case $y \in \operatorname{Fin} g$, from the regularity of g we get $r = 0$ and consequently $z = y$. Since f is a partial semiflow, an easy computation proves that $\omega_f(x) = s + \omega_f(z)$. Hence $t = s+u \leqslant s + \omega_f(z) = \omega_f(x)$, which means that $(t, x) \in \operatorname{dom} g\square(f\square g)$ and

$$\begin{aligned} \lim f\square(g\square f)(t_n, x_n) &= \lim f\big(t_n - s_n - r_n, g(r_n, \Omega_f(x_n))\big) \\ &= f(t-s, y) = f(t, x) = f\square(g\square f)(t, x). \end{aligned}$$

The proof is complete.

3. Boundary deformation retracts. For a subset $A \subset X$ we will denote $DA := \bar{A} \setminus A$

Definition 3.1. A subset $A \subset X$ is called *a boundary deformation retract* (BDR for short) if it satisfies the following two conditions

(3.1) $\bar{A}$ is a compact ANR,

(3.2) if DA is non-empty, then there exists an open neighbourhood U of DA and a continuous mapping $r\colon [0, 1] \times \bar{U} \to \bar{U}$ such that

$$\{r(t, x)\colon t \in [0, 1]\} \cap DA = \{r(1, x)\} \text{ for } x \in \bar{U}$$

and

$$r(t, x) = x \Leftrightarrow t = 0 \text{ or } x \in DA .$$

Observe that any compact ANR is obviously a BDR. Further examples of BDR's will be given in [5].

Definition 3.2. If A is a BDR, then *the modified Euler characteristic of A* is defined by the formula

$$\hat{\chi}(A) := \begin{cases} \chi(\bar{A}) - \chi(DA) & \text{if } DA \neq \phi \\ \chi(A) & \text{if } DA = \phi . \end{cases}$$

Observe that by (3.2) $r_t := r(t, \cdot)$ is a homotopy between the identity on $\bar{U}$ and a retraction $\bar{U} \to DA$. Hence, by known properties of ANR's (cf. [1], IV.3.2). DA is also a compact ANR. Thus Euler characteristics $\chi(\bar{A})$ and $\chi(DA)$ are defined (compare [6]) and $\hat{\chi}(A)$ is meaningful.

LEMMA 3.1. *Let $A \subset X$ be a BDR. Then there exists a sequence $\{g^{(n)}\}$ of regular deformations on $\bar{A}$ satisfying*

$$\mathrm{Fin}\, g^{(n)} = DA \qquad n = 1, 2, \ldots,$$

$$\exists \varepsilon > 0\colon\ \bar{A} \setminus \mathrm{Fix}_0\, g^{(n)} \subset K(DA, \varepsilon/n) . \tag{3.3}$$

$$\forall T > 0\colon\ \mathrm{Fix}_T\, g^{(n)} = \mathrm{Fix}_0\, g^{(n)}, \tag{3.4}$$

where $\mathrm{Fix}_0\, g^{(n)} := \bigcap_{T>0} \mathrm{Fix}_T\, g^{(n)}$. *Moreover* $\lim g^{(n)} = id$ *"uniformly" on $\bar{A}$ in the sense that*

$$\forall \varepsilon > 0\, \exists N \in \mathbf{N}\, \forall n \geq N\, \forall x \in \bar{A}\, \forall t \in I_{g^{(n)}}(x)\colon\ \varrho\big(g^{(n)}(t, x), x\big) \leq \varepsilon . \tag{3.5}$$

Proof. Changing U and r, if necessary, we may assume that $U = K(DA, \varepsilon)$ for some $\varepsilon > 0$. Set $U_n := K(DA, \varepsilon/n)$ and $W_n := \bar{A} \setminus U_n$. Let the deformation $g^{(n)}$ on $\bar{A}$ be given by:

$$g^{(n)}(t, x) := \begin{cases} r\left(t \dfrac{\varrho(x, W_n)}{\varrho(x, DA)}, x\right) & \text{for } x \in U_n \setminus DA,\ 0 \leq t\varrho(x, W_n) \leq \varrho(x, DA) \\ x & \text{for } x \in W_n,\ t \in \mathbf{R}^* \text{ or } x \in DA,\ t = 0 . \end{cases}$$

It is easy to see that the map $g^{(n)}$ is well defined and dom $g^{(n)}$ is closed. To prove its continuity, consider first a point $(0, x^*) \in \{0\} \times DA$. Define a multivalued mapping

$R\colon \bar{U} \to \bar{U}$ by $R(x) := \{r(s, x)\colon 0 \leqslant s \leqslant 1\}$. A compactness argument with the segment $[0, 1]$ shows that R is upper semi-continuous (i.e. $\forall x_0 \in \bar{U}\ \forall \varepsilon > 0\ \exists \delta > 0\colon x \in K(x_0, \delta) \cap \cap \bar{U} \Rightarrow R(x) \subset K(R(x_0), \varepsilon)$). Let $\varepsilon > 0$ be fixed. By the u.s.c. of R we may find $a\ \delta > 0$ such that $x \in K(x^*, \delta) \cap \bar{U} \Rightarrow R(x) \subset K(R(x^*), \varepsilon) = K(x^*, \varepsilon)$. In particular if $x \in K(x^*, \delta) \cap \bar{U}$ and $(t, x) \in \operatorname{dom} g^{(n)}$ then $g^{(n)}(t, x) \in K(x^*, \varepsilon)$, which proves the continuity of $g^{(n)}$ at $(0, x^*)$. The continuity at the remaining points is obvious.

From the condition (3.2) it follows immediately that $g^{(n)}$ is a regular deformation on $\bar{A}$, satisfying properties (3.3) and (3.4) and such that $\operatorname{Fin} g^{(n)} = DA$.

To prove the uniform convergence notice that

$$\lim_{n\to\infty} \sup\{\varrho(g^{(n)}(t, x), x)\colon\ x \in \bar{A},\ t \in I_{g^{(n)}}(x)\} \leqslant$$

$$\leqslant \lim_{n\to\infty} \sup\{\varrho(r(t, x), x)\colon\ x \in \bar{U}_n,\ t \in [0, 1]\}.$$

Hence, it is enough to prove that the second limit is zero. Assume the contrary. Then there is a sequence $\{(t_n, x_n)\} \subset [0, 1] \times \bar{U}_n$ such that $(t_n, x_n) \to (t, x) \in [0, 1] \times DA$ and $\varrho(r(t_n, x_n), x_n) \geqslant \varepsilon > 0$ for some $\varepsilon > 0$. Passing to the limit we obtain $\varrho(r(t, x), x) > 0$, which contradictes the fact, that $r(t, x) = x$ for $x \in DA$. The proof is complete.

4. Main results. Let f be a deformation on X and T a positive number. Define

$$\mathbf{Fix}_T f := f([0, T], \mathrm{Fix}_T f).$$

Lemma 4.1. *Let M be a compact ANR, T a positive number and f a deformation on M satisfying the condition*

$$\mathbf{Fix}_T f \cap \overline{\mathrm{Fin} f} = \phi. \tag{4.1}$$

Then $\mathrm{Fix}_T f \subset \operatorname{int} \operatorname{dom} f_T$.

Proof. Assume the contrary. We will find a sequence $\{x_n\}$, $x_n \to x \in \mathrm{Fix}_T f$ and such that for every $n \in N$ $x_n \notin \operatorname{dom} f_T$, i.e. $\omega_f(x_n) \leqslant T$. We may assume that there exists the limit $y = \lim \Omega_f(x_n)$. Obviously $y \in \overline{\mathrm{Fin} f}$, but also $y \in \mathbf{Fix}_T f$, which contradictes the assumption (4.1). The proof is finished.

Notice that if there exists a neighbourhood U of $\mathrm{Fix}_T f$ such that $U \subset \operatorname{dom} f_T$, then the fixed point index $i(f_T, U)$ is well defined. Moreover, its value does not depend on the choice of such a neighbourhood. Further on, we will denote the above common value by $i(f_T, M)$.

Theorem 4.1. *Let M be a compact ANR and f a regular deformation on M. Assume that $\mathrm{Fin} f$ is also a compact ANR and T is a positive number. If*

$$\mathbf{Fix}_T f \cap \mathrm{Fin} f = \phi, \tag{4.2}$$

then

$$i(f_T, M) = \chi(M) - \chi(\mathrm{Fin} f).$$

Proof. First notice that (4.2) implies (4.1). In fact, if there exists an $y \in \mathrm{Fix}_T f \cap \overline{\mathrm{Fin} f}$ then $y = f(t, x)$ for some $x \in \mathrm{Fix}_T f$, $t \leqslant T$. Since f is regular and $\mathrm{Fin} f = \overline{\mathrm{Fin} f}$, it must be $t = T$. Consequently $y = f(T, x) = x$, which contradictes the assumption (4.2).

Since $\mathrm{Fin} f$, as a compact ANR is also a BDR, we can find by Lemma 3.1 a regular deformation g on $\mathrm{Fin} f$ such that $\mathrm{Fin}\, g = \phi$. It means that $\mathrm{dom}\, g = \boldsymbol{R}^* \times \mathrm{Fin} f$. Consequently, $\mathrm{dom}\, g \Box f = \boldsymbol{R}^* \times M$. We will prove that

$$\mathrm{Fix}_T(g \Box f) = \mathrm{Fix}_T f \cup \mathrm{Fix}_T g\,.$$

Let $x \in \mathrm{Fix}_T(g \Box f)$. If $T \leqslant \omega_f(x)$, then $x = (g \Box f)(T, x) = f(T, x)$. Hence $x \in \mathrm{Fix}_T f$. In the opposite case $x = g\big(T - \omega_f(x), \Omega_f(x)\big)$, which means that $x \in \mathrm{Fin} f$. Thus, by the regularity of f, we obtain $x = g(T, x)$ and $x \in \mathrm{Fix}_T g$. The opposite inclusion is obvious.

By (4.2) there exist open, disjoint sets U, V such that $\mathrm{Fix}_T f \subset U$, $\mathrm{Fix}_T g \subset \mathrm{Fin} f \subset V$. Modifying U, if necessary, we may assume by Lemma 4.1 that $\overline{U} \subset \mathrm{dom} f_T$. Replacing V by $V \cap \omega_f^{-1}(0, T/2))$ we may also assume that $(g \Box f)_T(V) \subset \mathrm{Fin} f$. Now notice that the mapping $(g \Box f)_T$ is homotopic to the identity on M. Hence, from (i4) and the invariance of the Lefschetz number under homotopy we get $\chi(M) = \Lambda(\mathrm{id}_M) = \Lambda\big((g \Box f)_T\big) = i\big((g \Box f)_T, M\big)$. But, since $\overline{U} \subset \mathrm{dom} f_T$, we have $(g \Box f)_{T|\overline{U}} = f_{T|\overline{U}}$. Thus, by (i1)

$$i\big((g \Box f)_T, U\big) = i(f_T, U) = i(f_T, M)\,.$$

From (i5) it follows that

$$i\big((g \Box f)_T, V\big) = i\big((g \Box f)_{T|\mathrm{Fin} f}, V \cap \mathrm{Fin}\, f\big) = i(g_T, V \cap \mathrm{Fin}\, f) = \chi(\mathrm{Fin}\, f)\,.$$

Finally, from (i3) we get

$$\chi(M) = i\big((g \Box f)_T, M\big) = i\big((g \Box f)_T, U\big) + i\big((g \Box f)_T, V\big) = i(f_T, M) + \chi(\mathrm{Fin} f)\,.$$

The proof is finished.

Denote $\mathrm{Ret}\, f := D(\mathrm{Fin} f) = \overline{\mathrm{Fin} f} \setminus \mathrm{Fin} f$.

Definition 4.1. We will say that the deformation f satisfies *the property* (T) if

(T) $$\exists T_0 > 0\ \forall x \in \mathrm{Ret}\, f\ \ \omega_f(x) \geqslant T_0\,.$$

THEOREM 4.2. *Let M be a compact ANR, T a positive number and f a partial semiflow on M such that $\mathrm{Fin} f$ is a BDR. Assume f satisfies (T) and let*

(4.3) $$\mathrm{Fix}_t f \cap \overline{\mathrm{Fin} f} = \phi \quad \textit{for } 0 < t \leqslant T.$$

Then

(4.4) $$i(f_T, M) = \chi(M) - \chi(\mathrm{Fin} f)\,.$$

Proof. Let T_0 be a number appearing in (T). We will firstly consider the case $T \leqslant T_0$.

Let $\{g^{(n)}\}$ be a sequence of regular deformations on $\overline{\mathrm{Fin} f}$ defined as in Lemma 3.1. By Lemma 2.4, $f^{(n)} := f \Box (g^{(n)} \Box f)$ is a deformation. We will prove that there exists an $n \in \boldsymbol{N}$, such that

(4.5) $$\mathrm{Fix}_T f^{(n)} = \mathrm{Fix}_T f \cup \mathrm{Fix}_T g^{(n)}.$$