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Mixed derivatives in the systems of differential equations

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§ 1. In this paper we shall consider the system

(1.1) $$\frac{\partial u_l}{\partial t} = f_l\left((t, x, u, \frac{\partial u_l}{\partial x}, \frac{\partial^2 u_l}{\partial x^2}\right) \quad (l = 1, 2, \ldots, n),$$

where $x = (x_1, x_2, \ldots, x_p)$, $\frac{\partial u_l}{\partial x}$ denotes the vector of derivatives of the first order and $\frac{\partial^2 u_l}{\partial x^2}$ is the $p \times p$ matrix $\left(\frac{\partial^2 u_l}{\partial x_i \partial x_j}\right)$ $(i, j = 1, 2, \ldots, p)$ $(l = 1, \ldots, n)$ of the derivatives of the second order. In the system (1.1) the l-th equation contains all the functions $u_1, u_2, \ldots$ $\ldots, u_n$ and the derivatives of the l-th function $u_l(t, x)$ only.

We do not suppose that the metrix $\left(\frac{\partial f_l}{\partial q_{ij}}\right)$ $(i, j = 1, \ldots, p)$ possesses the dominating diagonal line. Also we do not use the "small" difference expressions of the second order depending on the sign of the derivatives $\frac{\partial f_l}{\partial q_{ij}}$, a detail which seems to be disappointing in the construction of the difference scheme.

We shall use the "large" difference expressions for the mixed derivatives, cf. the formula (2.9) and Fig. 1.

The term

(1.2) $$\frac{1}{p^2 - p} \cdot \sum_{\substack{i=1 \\ i \neq j}}^{p} \sum_{j=j}^{p} \frac{1}{3} (v_l^{-ij(M)} + v_l^M + v_l^{i-j(M)}),$$

can be found twice, first on the left-hand side of the difference equation (2.5) and then in the formula (3.4) and represents the arithmetic mean of the values

(1.3) $$v_l^{-ij(M)}, v_l^M, v_l^{i-j(M)},$$

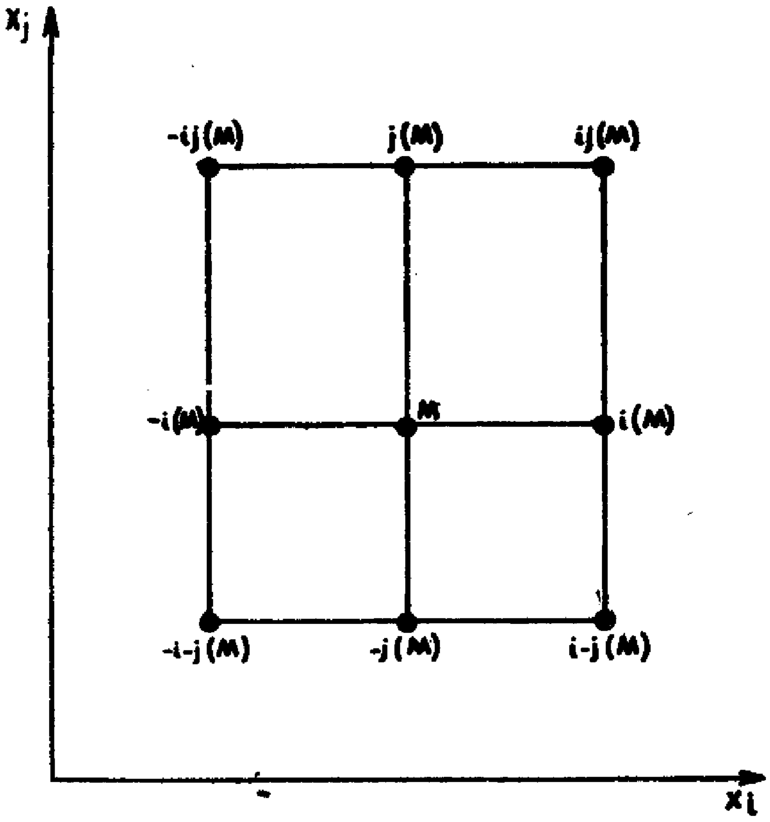

Fig. 1. The nodal points with indices M, $ij(M), \ldots, -i-j(M)$

which enter into the difference expressions of the second order (2.8) and (2.9) with negative signs, cf. Fig. 1 and Fig. 2.

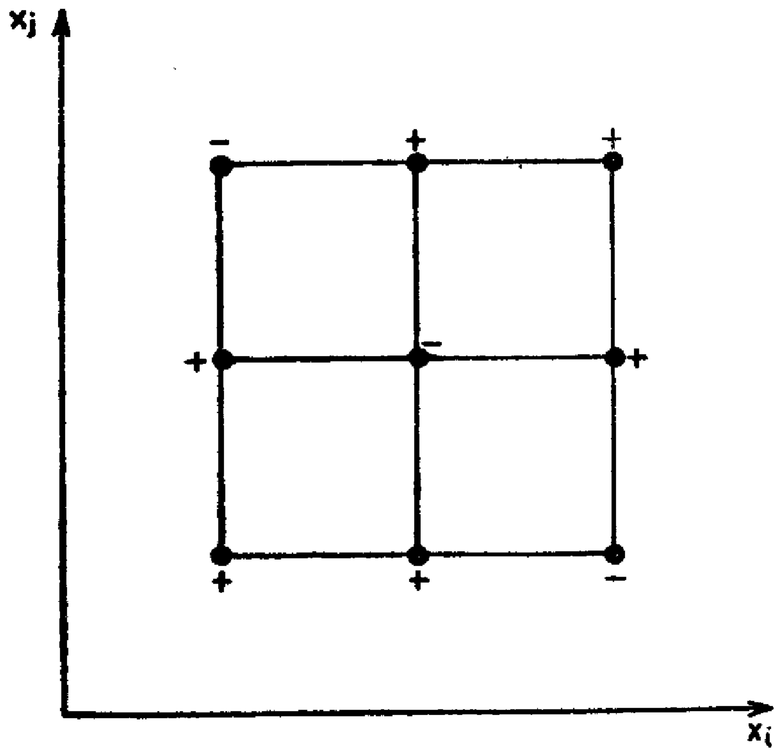

Fig. 2. The signs of the corresponding terms in the difference expressions of the second order

It turns out as can be seen at appropriate places that these negative signs could deteriorate the proof of the Theorem 1, cf. the formula (3.13a). The resulting difficulties can be removed by a suitable choice of the time interval k, cf. the formula (3.15) and (3.16).

§ 2. We shall assume that the functions $f_l(t, x, u, \overset{1}{q}, \overset{2}{q})$, $x = (x_1, \ldots, x_p)$, $u = (u_1, \ldots, u_n)$, $\overset{1}{q} = (\overset{1}{q}_1, \ldots, \overset{1}{q}_p)$, $\overset{2}{q} = (q_{ij})$ $(i, j = 1, \ldots, p)$ are of the class C^1 in the set $\mathscr{D}_1$: $0 \leqslant t \leqslant T$, $0 \leqslant x_j \leqslant \alpha$, $-\infty < u_l < +\infty$, $-\infty < \overset{1}{q}_j < +\infty$, $-\infty < q_{ij} < +\infty$ $(l = 1, \ldots, n)$ $(i, j = 1, \ldots, p)$.

We consider the following boundary problem in the set $\mathscr{D}$: $0 \leqslant t \leqslant T$, $0 \leqslant x_j \leqslant \alpha$ $(j = 1, \ldots, p)$:

$$\frac{\partial u_l}{\partial t} = f_l\left(t, x, u, \frac{\partial u_l}{\partial x}, \frac{\partial^2 u_l}{\partial x^2}\right) \quad (l = 1, 2, \ldots, n), \tag{2.1}$$

$$\begin{cases} u_l(0, x) = \varphi_{l0}(x), \\ u_l(t, x) = \varphi_{lj}(t, x), & \text{for } x_j = 0, \\ u_l(t, x) = \psi_{lj}(t, x), & \text{for } x_j = \alpha, \\ (l = 1, \ldots, n)(j = 1, \ldots, p). \end{cases} \tag{2.2}$$

In the equation (2.1) $\frac{\partial u_l}{\partial x}$ denotes the vector

$$\frac{\partial u_l}{\partial x} = \left(\frac{\partial u_l}{\partial x_1}, \frac{\partial u_l}{\partial x_2}, \ldots, \frac{\partial u_l}{\partial x_p}\right)$$

and $\frac{\partial^2 u_l}{\partial x^2}$ is the $p \times p$ matrix of the partial derivatives

$$\frac{\partial^2 u_l}{\partial x^2} = \left(\frac{\partial^2 u_l}{\partial x_i \partial x_j}\right) \; (i, j = 1, \ldots, p)(l = 1, \ldots, n).$$

We shall assume that the solution $u(t, x)$ of the problem (2.1), (2.2) exists and is of the class C^2 in the set $\mathscr{D}$.

We assume also that

$$\left|\frac{\partial f_l}{\partial u_\lambda}\right| \leqslant \mathscr{L}, \left|\frac{\partial f_l}{\partial \overset{1}{q}_j}\right| \leqslant \Gamma_j \; (j = 1, \ldots, p) \tag{2.3}$$

$(l, \lambda = 1, \ldots, n)$

and

$$0 < g_{ij} \leqslant \frac{\partial f_l}{\partial q_{ij}} \leqslant \mathscr{G}_{ij} \quad (i, j = 1, \ldots, p)(l = 1, \ldots, n), \tag{2.4}$$

in the set $\mathscr{D}_1$.

The corresponding difference equation is of the explicite type and will be written in the following form

$$\frac{1}{k} \cdot \left[v_l^{\omega(M)} - \frac{1}{p^2 - p} \cdot \sum_{i=1}^{p} \sum_{\substack{j=1 \\ i \neq j}}^{p} \frac{1}{3} \left(v_l^{-ijM)} + v_l^M + v_l^{i-j(M)}\right) = \right. \tag{2.5}$$

$$= f_l(t^\mu, x^m, v^M, v_l^{M\overline{1}}, v_l^{M\overline{2}}) \quad (l = 1, \ldots, n),$$

cf. Fig. 3.

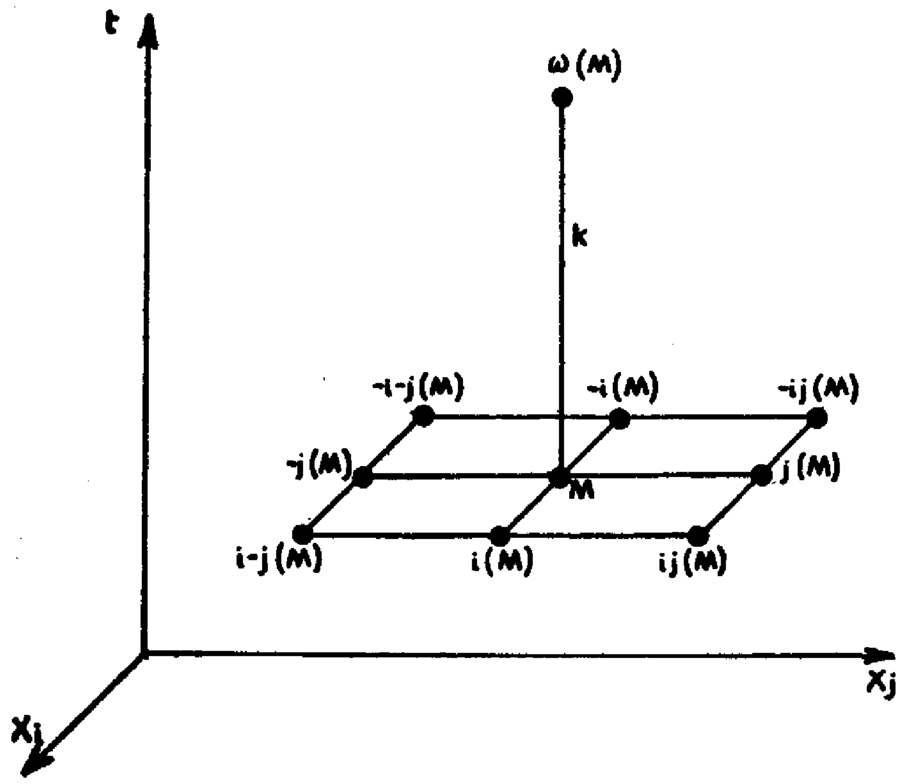

Fig. 3. The nodal points with indices $\omega(M)$, M, $ij(M), \ldots, -i-j(M)$

Here we use the notation $M = (\mu, m)$ of the paper [1], $v_l^{M\bar{1}}$ denotes the vector of the symmetric difference quotients of the first order

(2.6) $$v_l^{M\bar{1}} = (v_l^{M1}, v_l^{M2}, \ldots, v_l^{Mp}) \quad (l = 1, \ldots, n),$$

where

(2.7) $$v_l^{Mj} = \frac{1}{2h} \cdot (v_l^{j(M)} - v_l^{-j(M)}) \quad (l = 1, \ldots, n),$$

and $v_l^{M\bar{2}}$ the $p \times p$ matrix $v_l^{M\bar{2}} = (v_l^{Mij})$ $(i, j = 1, \ldots, p)(l = 1, \ldots, n)$ whose elements are

(2.8) $$\begin{cases} v_l^{Mjj} = \dfrac{1}{h^2} \cdot (v_l^{j(M)} - 2v_l^M + v_l^{-j(M)}), \\ (l = 1, \ldots, n)(j = 1, \ldots, p), \end{cases}$$

and

(2.9) $$\begin{cases} v_l^{Mij} = \dfrac{1}{4h^2} \cdot (v_l^{ij(M)} - v_l^{-ij(M)} - v_l^{i-j(M)} + v_l^{-i-j(M)}), \\ (i \neq j)(i, j = 1, \ldots, p)(l = 1, \ldots, n), \end{cases}$$

respectively.

The boundary conditions are induced by the boundary conditions (2.2) and have the form

(2.10) $$\begin{cases} v_l^M = \varphi_{l0}(x^m), & \text{for } M = (0, m), \\ v_l^M = \varphi_{lj}(t^\mu, x^m), & \text{for } m_j = 0, \\ v_l^M = \psi_{lj}(t^\mu, x^m), & \text{for } m_j = N, \\ (j = 1, \ldots, p)(l = 1, \ldots, n), \end{cases}$$

where $hN = \alpha$.

The mesh size h for the space coordinates x_j ($j = 1, \dots, p$) and k for the time coordinate t satisfy the conditions

$$(2.11)\qquad g_{jj}\cdot\frac{1}{h}-\Gamma_j\cdot\frac{1}{2}\geqslant 0 \quad (j=1,\dots,p),$$

and

$$(2.12)\qquad \frac{1}{3k}\cdot\frac{1}{p^2-p}-\frac{1}{4h^2}\cdot\mathscr{G}_{ij}\geqslant 0 \quad (i\neq j),$$

$$(2.13)\qquad \frac{1}{3k}\cdot\frac{1}{p^2-p}-\frac{2}{h^2}\cdot\sum_{j=1}^{p}\mathscr{G}_{jj}\geqslant 0,$$

for $i, j = 1, \dots p$.

This means that

$$(2.15)\qquad k\leqslant 4h^2\cdot\frac{1}{p^2-p}\cdot\frac{1}{3}\cdot\frac{1}{\mathscr{G}_{ij}} \quad (i\neq j),$$

and

$$(2.16)\qquad k\leqslant h^2\cdot\frac{1}{3}\cdot\frac{1}{2\sum_{j=1}^{p}\mathscr{G}_{jj}},$$

for $i, j = 1, \dots, p$.

We define the error η_l^M ($l = 1, \dots, n$) by

$$(2.17)\qquad \frac{1}{k}\cdot\left[u_l^{\omega(M)}-\frac{1}{p^2-p}\cdot\sum_{\substack{i=1\\ i\neq j}}^{p}\sum_{j=1}^{p}\frac{1}{3}(u_l^{-ij(M)}+u_l^M+u_l^{i-j(M)})\right]=$$
$$= f_l(t^\mu, x^m, u^M, u_l^{M\bar{1}}, u_l^{M\bar{2}})+\eta_l^M, \qquad (l=1,\dots,n),$$

and we have

$$(2.18)\qquad \varepsilon(h,k)\to 0, \quad \text{as } h,k\to 0,$$

where

$$(2.19)\qquad \begin{cases}\varepsilon(h,k)=\max\limits_{l}\varepsilon_l(h,k) & (l=1,\dots,n),\\ \varepsilon_l(h,k)=\max\limits_{M}|\eta_l^M|.\end{cases}$$

(2.18) means that the system of difference equations (2.5) is consistent with the system of differential equations (2.1).

We define also the error

$$(2.20)\qquad r_l^M = u_l^M - v_l^M \quad (l=1,\dots,n).$$

§ 3. THEOREM 1. *Under the assumption of* § 2 *the difference method is convergent.*

Proof. We shall introduce the maximal values

$$(3.1)\qquad s_l^{\mu\sim} = \max_m r_l^{\mu,m} = r_l^{\mu,b(l)} = r_l^{B(l)},$$

$$(3.2)\qquad s_l^{\mu+1} = \max_m r_l^{\mu+1,m} = r_l^{\mu+1,a(l)} = r_l^{\omega(A(l))},$$

for $l = 1, \ldots, n$, where the nodal points $B(l)$ and $A(l)$ depend on the number l.
We can write

$$(3.3)\qquad s_l^{\mu\sim} = \frac{1}{k}(s_l^{\mu+1} - s_l^{\mu}) = \frac{1}{k}(r_l^{\omega(A(l))} - r_l^{B(l)}),$$

or

$$(3.4)\qquad s_l^{\mu\sim} = \frac{1}{k}\cdot\left[r_l^{\omega(A(l))} - \frac{1}{p^2-p}\cdot\sum_{\substack{i=1\\ i\neq j}}^{p}\sum_{j=1}^{p}\frac{1}{3}(r_l^{-ij(A(l))} + r_l^{A(l)} + r_l^{i-j(A(l))})\right] +$$
$$+\frac{1}{k}\cdot\left[\frac{1}{p^2-p}\cdot\sum_{\substack{i=1\\ i\neq j}}^{p}\sum_{j=1}^{p}\frac{1}{3}(r_l^{-ij(A(l))} + r_l^{A(l)} + r_l^{i-j(A(l))}) - r_l^{B(l)}\right].$$

The first square bracket in (3.4) can be calculated with the aid of the equations (2.17) and (2.5). To this end we subtract the equations (2.17) and (2.5), we apply the mean value theorem and we get

$$(3.5)\qquad s_l^{\mu\sim} = \eta_l^{A(l)} + \sum_{\lambda=1}^{n}\frac{\partial f_l}{\partial u_\lambda}(\sim)\cdot r_\lambda^{A(l)} + \sum_{j=1}^{p}\frac{\partial f_l}{\partial \overset{1}{q}_j}(\sim)\cdot\frac{1}{2h}\cdot(r^{j(A(l))} - r_l^{-j(A(l))}) +$$
$$+\sum_{j=1}^{p}\frac{\partial f_l}{\partial q_{jj}}(\sim)\cdot\frac{1}{h^2}\cdot(r_l^{j(A(l))} - 2r_l^{A(l)} + r_l^{-j(A(l))}) +$$
$$+\sum_{\substack{i=1\\ i\neq j}}^{p}\sum_{j=1}^{p}\frac{\partial f_l}{\partial q_{ij}}(\sim)\cdot\frac{1}{4h^2}\cdot(r_l^{ij(A(l))} - r_l^{-ij(A(l))} - r_l^{i-j(A(l))} + r_l^{-i-j(A(l))}) +$$
$$+\frac{1}{k}\left[\frac{1}{p^2-p}\cdot\sum_{\substack{i=1\\ i\neq j}}^{p}\sum_{j=1}^{p}\frac{1}{3}(r_l^{-ij(A(l))} + r_l^{A(l)} + r_l^{i-j(A(l))}) - r_l^{B(l)}\right].$$

We can now introduce $r_l^{B(l)}$ at suitable places and we obtain

$$(3.6)\qquad s_l^{\mu\sim} = \eta_l^{A(l)} + \sum_{\lambda=1}^{n}\frac{\partial f_l}{\partial u_\lambda}(\sim)\cdot r_l^{A(l)} + \mathscr{C}_{l1} + \mathscr{C}_{l2}, \qquad (l = 1, \ldots, n),$$

where

$$(3.7)\qquad \mathscr{C}_{l1}=\sum_{i=1}^{p}\sum_{\substack{j=1\\ i\neq j}}^{p}\frac{\partial f_l}{\partial q_{ij}}(\sim)\cdot\frac{1}{4h^2}\cdot[(r_l^{ij(A(l))}-r_l^{B(l)})+(r_l^{-i-j(A(l)}-r_l^{B(l)})]+$$

$$+\sum_{j=1}^{p}\frac{\partial f_l}{\partial q_{jj}}(\sim)\cdot\frac{1}{h^2}[(r_l^{j(A(l))}-r_l^{B(l)})+(r_l^{-j(A(l))}-r_l^{B(l)})]+$$

$$+\sum_{j=1}^{p}\frac{\partial f_l}{\partial \overset{1}{q}_j}(\sim)\cdot\frac{1}{2h}\cdot[(r_l^{j(A(l))}-r_l^{B(l)})-(r_l^{-j(A(l))}-r_l^{B(l)})],$$

and

$$(3.8)\qquad \mathscr{C}_{l2}=\sum_{i=1}^{p}\sum_{\substack{j=1\\ i\neq j}}^{p}\frac{\partial f_l}{\partial q_{ij}}(\sim)\cdot\frac{1}{4h^2}\cdot[-(r_l^{-ij(A(l))}-r_l^{B(l)})+(r_l^{i-j(A(l))}-r_l^{B(l)})]+$$

$$+\sum_{j=1}^{p}\frac{\partial f_l}{\partial q_{jj}}(\sim)\cdot\frac{1}{h^2}\cdot[-2(r_l^{A(l)}-r_l^{B(l)})]+$$

$$+\frac{1}{k}\cdot\frac{1}{p^2-p}\cdot\sum_{i=1}^{p}\sum_{\substack{j=1\\ i\neq j}}^{p}\frac{1}{3}\cdot[(r_l^{-ij(A(l))}-r_l^{B(l)})+(r_l^{A(l)}-r_l^{B(l)})+$$

$$+(r_l^{i-j(A(l))}-r_l^{B(l)})].$$

In the last line of the formula (3.5) we have substitute

$$(3.9)\qquad r_l^{B(l)}=\frac{1}{p^2-p}\cdot\sum_{i=1}^{p}\sum_{\substack{j=1\\ i\neq j}}^{p}r_l^{B(l)},$$

and we have write

$$(3.10)\qquad \frac{1}{k}\cdot\left[\frac{1}{p^2-p}\cdot\sum_{i=1}^{p}\sum_{\substack{j=1\\ i\neq j}}^{p}\frac{1}{3}(r_l^{-ij(A(l))}+r_l^{A(l)}+r_l^{i-j(A(l))})-r_l^{B(l)}\right]=$$

$$=\frac{1}{k}\cdot\frac{1}{p^2-p}\cdot\sum_{i=1}^{p}\sum_{\substack{j=1\\ i\neq j}}^{p}\frac{1}{3}[(r_l^{-ij(A(l))}-r_l^{B(l)})+(r_l^{A(l)}-r_l^{B(l)})+$$

$$+(r_l^{i-j(A(l))}-r_l^{B(l)})].$$

In $\mathscr{C}_{l1}$, cf. (3.7), we have collected the terms corresponding to the nodal points with indices $ij(A(l))$, $-i-j(A(l))$, $j(A(l))$, $-j(A(l))$. These terms enter into difference expressions of the second order with the signs $+$, cf. Fig. 2, and are easy to handle. In fact, the first line in (3.7) is non-positive and can be dropped. The second and third line in (3.7) can be rewritten as follows

$$(3.11)\qquad \sum_{j=1}^{p} \frac{\partial f_l}{\partial q_{ij}}(\sim)\cdot\frac{1}{h^2}\left[(r_l^{j(A(l))}-r_l^{B(l)})+(r_l^{-j(A(l))}-r_l^{B(l)})\right]+$$

$$+\sum_{j=1}^{p} \frac{\partial f_l}{\partial \overset{1}{q}_j}(\sim)\cdot\frac{1}{2h}\left[(r_l^{j(A(l))}-r_l^{B(l)})-(r_l^{-j(A(l))}-r_l^{B(l)})\right] =$$

$$=\sum_{j=1}^{p} (r_l^{j(A(l))}-r_l^{B(l)})\cdot\left(\frac{\partial f_l}{\partial q_{jj}}(\sim)\cdot\frac{1}{h^2}+\frac{\partial f_l}{\partial \overset{1}{q}_j}(\sim)\cdot\frac{1}{2h}\right)+$$

$$+(r_l^{-j(A(l))}-r_l^{B(l)})\cdot\left(\frac{\partial f_l}{\partial q_{jj}}(\sim)\cdot\frac{1}{h^2}-\frac{\partial f_l}{\partial \overset{1}{q}_j}(\sim)\cdot\frac{1}{2h}\right).$$

But we have

$$(3.12)\qquad \frac{\partial f_l}{\partial q_{jj}}(\sim)\cdot\frac{1}{h^2}+\frac{\partial f_l}{\partial \overset{1}{q}_j}(\sim)\cdot\frac{1}{2h}\geqslant\frac{1}{h}\left(g_{jj}\cdot\frac{1}{h}-\Gamma_j\cdot\frac{1}{2}\right)\geqslant 0\,,$$

$$(3.13)\qquad \frac{\partial f_l}{\partial q_{jj}}(\sim)\cdot\frac{1}{h^2}-\frac{\partial f_l}{\partial \overset{1}{q}_j}(\sim)\cdot\frac{1}{2h}\geqslant\frac{1}{h}\left(g_{jj}\cdot\frac{1}{h}-\Gamma_j\cdot\frac{1}{2}\right)\geqslant 0\,,$$

because of the assumption (2.11), hence the right-hand side of the formula (3.11) is non-positive and can be dropped also.

The problem arises in (3.8) with terms corresponding to the nodal points with indices $-ij(A(l))$, $A(l)$, $i-j(A(l))$. These terms enter into the difference expressions of the second order with the negative signs $-$, cf. Fig. 2, and must be treated with the greater care. For example in the second line of the formula (3.8) we have

$$(3.13a)\qquad \frac{\partial f_l}{\partial q_{ij}}(\sim)\cdot\frac{1}{4h^2}\cdot\left[-(r_l^{-ij(A(l))}-r_l^{B(l)})\right]\geqslant 0\,,$$

and these terms cannot be dropped since they are non-negative.

But we have

$$(3.14)\qquad \mathscr{C}_{l2}=\sum_{i=1}^{p}\sum_{\substack{j=1\\ i\neq j}}^{p}(r_l^{-ij(A(l))}-r_l^{B(l)})\cdot\left(\frac{\partial f_l}{\partial q_{ij}}(\sim)\cdot\frac{-1}{4h^2}+\frac{1}{k}\cdot\frac{1}{p^2-p}\cdot\frac{1}{3}\right)+$$

$$+\sum_{\substack{i=1}}^{p}\sum_{\substack{j=1\\ i\neq j}}^{p}(r_l^{i-j(A(l))}-r_l^{B(l)})\cdot\left(\frac{\partial f_l}{\partial q_{ij}}(\sim)\cdot\frac{-1}{4h^2}+\frac{1}{k}\cdot\frac{1}{p^2-p}\cdot\frac{1}{3}\right)+$$

$$+(r_l^{A(l)}-r_l^{B(l)})\cdot\left(\sum_{j=1}^{p}\frac{\partial f_l}{\partial q_{jj}}(\sim)\cdot\frac{-2}{h^2}+\frac{1}{k}\cdot\frac{1}{p^2-p}\cdot\sum_{i=1}^{p}\sum_{\substack{j=1\\ i\neq j}}^{p}\frac{1}{3}\right).$$

In the first and second line of the formula (3.14) we have

$$\frac{\partial f_l}{\partial q_{ij}}(\sim)\cdot\frac{-1}{4h^2}+\frac{1}{k}\cdot\frac{1}{p^2-p}\cdot\frac{1}{3}\geqslant\frac{-1}{4h^2}\cdot\mathscr{G}_{ij}+\frac{1}{k}\cdot\frac{1}{p^2-p}\cdot\frac{1}{3}\geqslant 0\,, \tag{3.15}$$

because of the assumption (2.12), which means that the first and second line in (3.14) are non-positive and can be dropped.

In the third line of the formula (3.14) we have

$$\sum_{j=1}^{p}\frac{\partial f_l}{\partial q_{jj}}(\sim)\cdot\frac{-2}{h^2}+\frac{1}{k}\cdot\frac{1}{p^2-p}\cdot\sum_{i=1}^{p}\sum_{\substack{j=1\\ i\neq j}}^{p}\frac{1}{3}\geqslant\frac{-2}{h^2}\cdot\sum_{j=1}^{p}\mathscr{G}_{jj}+\frac{1}{k}\cdot\frac{1}{3}\geqslant 0\,, \tag{3.16}$$

because of the assumption (2.13). Hence, the third line in (3.14) is non-positive and can be dropped also.

Thus (3.7) and (3.8) reduces to

$$\mathscr{C}_{l1}\leqslant 0\,,\qquad \mathscr{C}_{l2}\leqslant 0\,, \tag{3.17}$$

and we can majorize the right-hand member in (3.6) and write:

$$s_l^{\mu\sim}\leqslant\eta_l^{A(l)}+\sum_{\lambda=1}^{n}\frac{\partial f_l}{\partial u_\lambda}(\sim)\cdot r_\lambda^{A(l)}\qquad (l=1,2,\ldots,n)\,. \tag{3.18}$$

In a similar way we can introduce the minimum values

$$z_l^{\mu}=\min_m r_l^{\mu,m}=r_l^{\mu,d(l)}=r_l^{\mathscr{D}(l)}\,, \tag{3.19}$$

$$z_l^{\mu+1}=\min_m r_l^{\mu+1,m}=r_l^{\mu+1,c(l)}=r_l^{\omega(\mathscr{C}(l))}\,, \tag{3.20}$$

for $l=1,2,\ldots,n$, where the nodal points $\mathscr{D}(l)$ and $\mathscr{C}(l)$ depend on the number l, and we obtain the difference inequalities

$$\begin{cases} z_l^{\mu\sim}\geqslant\eta_i^{\mathscr{C}(l)}+\displaystyle\sum_{\lambda=1}^{n}\frac{\partial f_l}{\partial u_\lambda}(\sim)\cdot r_l^{\mathscr{C}(l)} \\ (l=1,2,\ldots,n)\,. \end{cases} \tag{3.21}$$

From (3.21) and (3.18) it follows that

(3.22) $$s_l^{\mu\sim} \leqslant \mathscr{L}\cdot \sum_{\lambda=1}^{n} |r_\lambda^{A(l)}| + \varepsilon(h, k),$$

and

(3.23) $$z_l^{\mu\sim} \geqslant -\mathscr{L}\cdot \sum_{\lambda=1}^{n} |r_l^{\mathscr{C}(l)}| - \varepsilon(h, k).$$

We shall proceed now as in the paper [2], cf. [2] Lemma 3 and Lemma 4.

We define first

(3.24) $$R^\mu = \max_l \max_m |r_l^M|, \quad \text{for } M = (\mu, m),$$

and obtain

(3.25) $$(\max_m |r_l^M|)^\sim \leqslant \max(s_l^{\mu\sim}, -z_l^{\mu\sim}), \quad \text{for } M = (\mu, m) \quad (l = 1, \ldots, n)$$

because of Lemma 3. Then we can apply Lemma 4 and we get

(3.26) $$R^{\mu\sim} \leqslant n\cdot\mathscr{L}\cdot R^\mu + \varepsilon(h, k), \; R^0 = 0,$$

for $\mu = 0, 1, \ldots, N_1$, where $k\cdot N_1 = T$.

This yields

(3.27) $$R^\mu \leqslant \frac{\varepsilon(h, k)}{n\cdot\mathscr{L}}\cdot(e^{n\mathscr{L}k\mu} - 1)(\mu = 0, 1, \ldots, N_1).$$

But $|r_l^M| \leqslant R^\mu$ for $M = (\mu, m)$ because of the definition (3.24), hence we have the error estimate

(3.28) $$|r_l^M| \leqslant \frac{\varepsilon(h, k)}{n\cdot\mathscr{L}}(e^{n\mathscr{L}k\mu} - 1)(\mu = 0, 1, \ldots, N_1).$$

The convergence of the difference method follows from the error estimate (3.28) and the condition (2.18).

This completes the proof of Theorem 1.

References

[1] Z. Kowalski, *A difference method for a non-linear parabolic differential equation without mixed derivatives*, Ann. Polon. Math., 20 (1968), 167—177.

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