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A stability-like condition in generalized dynamical systems

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1. Stability-like concepts in various versions, have been considered by several authors. The purpose of the present paper is to follow the fundamental idea of Lyapunov stability and to extend it in some sense with respect to "multivalued" (dynamical) systems, called often generalized systems or generalized pseudo-dynamical semi-systems. The theory of such systems is already developed; we limit ourself to refer for instance the papers [6] and [7].

For informations on usual dynamical systems we refer to the book [1] where a rich bibliography can be found, for certain generalizations of them (semi-systems etc.) to [3], [4]. For our purpose we will adopt a simplified terminology, introducing the names: a system and a generalized system. Notice only that systems in the sense defined below are semi-systems or pseudo dynamical systems in other papers (cf. [4] or [3]); there are also another names in the mathematical literature.

A triple $(X, G; \pi)$, where X is a non-empty set (called the space), $(G, +)$ is an abelian semi-group with a neutral element 0 and π is a mapping from $G \times X$ into X, is said to be a system iff:

(1.1) $$\pi(0, x) = x \quad \text{for every } x \in X$$

(1.2) $$\pi(t, \pi(s, x)) = \pi(t+s, x) \quad \text{for } t, s \in G,\ x \in X.$$

Denote by $\mathscr{P}(Y)$ the family of all non-empty subsets of the set Y.

A triple $(X, G; \lambda)$, where X and G are as above, and λ is a mapping from $G \times X$ into $\mathscr{P}(X)$, is said to be a generalized system, iff

(1.3) $$\lambda(0, x) = \{x\} \quad \text{for } x \in X$$

(1.4) $$\lambda(s, \lambda(t, x)) \subset \lambda(t+s, x) \quad \text{for } t, s \in G,\ x \in X,$$

where a natural notation

(1.5) $$\lambda(s, A) := \bigcup \{\lambda(s, y) \colon y \in A\} \quad \text{for } s \in G,\ A \in \mathscr{P}(X)$$

is used.

If instead of (1.4) we have

(1.6) $$\lambda(s, \lambda(t, x)) = \lambda(t+s, x) \quad \text{for } t, s \in G,\ x \in X,$$

then the system $(X, G; \lambda)$ is called regular.

It is known that fundamental examples of systems $(X, G; \pi)$ are given by solutions of autonomous differential equations with the uniqueness property. If we omit the assumption on the uniqueness, then we obtain generalized systems (see Sec. 4 in the sequel).

A mapping $\beta \colon M \to \mathscr{P}(\mathscr{P}(X))$, where $M \in \mathscr{P}(X)$, is said to be normal iff for every $x \in M$ we have:

$$x \in B \quad \text{for every } B \in \beta(x).$$

In order to explain briefly the main idea of the paper let us consider a special case: X being a metric space (with a metric ϱ) and $G = \boldsymbol{R}_+ := [0, \infty)$. Let M be a non-empty subset of X. Put $B(M, \varepsilon) := \{y \in X \colon d(y, M) < \varepsilon\}$, where $d(y, M) := \inf\{\varrho(x, y) \colon x \in M\}$, and $B(x, \varepsilon) := B(\{x\}, \varepsilon)$ for $x \in X$. The set M is said to be Lyapunov stable in the system $(X, \boldsymbol{R}_+; \pi)$ iff

$$\forall_{x \in M} \forall_{\varepsilon > 0} \exists_{\delta > 0} \big(\pi(B(x, \delta)) \subset B(M, \varepsilon)\big),$$

where $\pi(B(x, \delta)) := \{\pi(t, y) \colon t \in \boldsymbol{R}_+, y \in B(x, \delta)\}$.

This condition can be extended with respect to generalized systems $(X, \boldsymbol{R}_+; \lambda)$, in two natural ways:

(I) $$\forall_{x \in M} \forall_{\varepsilon > 0} \exists_{\delta > 0} \big(\lambda(B(x, \delta)) \subset B(M, \varepsilon)\big)$$

(the condition in brackets can be written also in the form

$$\lambda(t, y) \subset B(M, \varepsilon) \quad \text{for every } t \in \boldsymbol{R}_+ \text{ and } y \in B(x, \delta)),$$

(II) $$\forall_{x \in M} \forall_{\varepsilon > 0} \exists_{\delta > 0} \big(\lambda(t, y) \cap B(M, \varepsilon) \neq \emptyset \quad \text{for } t \in \boldsymbol{R}_+,\ y \in B(x, \delta)\big).$$

It is clear that (I) $\Rightarrow$ (II) but not conversely.

Fundamental theorems on stability conditions are stated with respect to so-called Lyapunov functions. A general definition of a Lyapunov function in (general) systems is proposed in [3] (see also [4]), and theorems giving sufficient and necessary conditions for a set M to be stable by means of such Lyapunov functions are proved as well.

These concepts and theorems can be extended for generalized systems and stabilities of the type (I). It seems to be impossible to get the same results with respect to (II). We shall propose some stability-like concept (between (I) and (II)) for which the idea of using Lyapunov functions in order to obtain sufficient and necessary conditions for stability is working similarly as for (I).

2. Let $(X, G; \lambda)$ be a generalized system and let $M \in \mathscr{P}(X)$ be given. Suppose that there are: a non-empty subfamily Ω of $\mathscr{P}(X)$ an a normal mapping $\beta \colon M \to \mathscr{P}(\mathscr{P}(X))$.

Definition 2.1. Let $A \in \mathscr{P}(X)$ and $n \in N$ be fixed. We define $P_\lambda(A, n) \subset X$ by the formula

$$(2.1)\qquad \begin{cases} y \in P_\lambda(A, n) \overset{\mathrm{df}}{\Leftrightarrow} \\ \forall_{t_0 \in G} \exists_{z_1 \in \lambda(t_0, y)} \forall_{t_1 \in G} \exists_{z_2 \in \lambda(t_1, z_1)} \ldots \\ \forall_{t_{n-1} \in G} \exists_{z_n \in \lambda(t_{n-1}, z_{n-1})} \forall_{t_n \in G} \; ((\lambda(t_n, z_n) \cap A \neq \emptyset) \end{cases}$$

Remark 2.1. It is clear that $y \in P_\lambda(A, n)$ if and only if

$$\forall_{t_0 \in G} \exists_{z_1 \in \lambda(t_0, y)} \forall_{t_1 \in G} \exists_{z_2 \in \lambda(t_1, z_1)} \cdots \forall_{t_{n-1} \in G} \exists_{z_n \in \lambda(t_{n-1}\; z_{n-1})} \forall_{t_n \in G} \exists_{z_{n+1} \in \lambda(t_n, z_n)} (z_{n+1} \in A) .$$

Definition 2.2. For $A \in \mathscr{P}(X)$ we put

$$(2.2)\qquad E_\lambda(A) := \bigcap \{P_\lambda(A, n) \colon n \in N\} .$$

If λ is fixed (as for instance throughout this section), then we shall write shortly $P(A, n)$ and $E(A)$ instead of $P_\lambda(A, n)$ and $E_\lambda(A)$, respectively.

PROPOSITION 2.1. *For every $A \in \mathscr{P}(X)$ and every $n, m \in N$ such that $m \leqslant n$, the inclusions*

$$(2.3)\qquad P(A, n) \subset P(A, m) \subset A$$

hold true.

Proof. Let $n = m+p$ with some (fixed) $p \in \{0\} \cup N$. We have

$$(2.4)\qquad \begin{cases} y \in P(A, n) \Leftrightarrow \forall_{t_0 \in G} \exists_{z_1 \in \lambda(t_0, y)} \forall_{t_1 \in G} \exists_{z_2 \in \lambda(t_1, z_1)} \cdots \\ \ldots \forall_{t_m} \exists_{z_{m+1} \in \lambda(t_m, z_m)} \cdots \\ \ldots \forall_{t_{m+p}} \exists_{z_{m+p+1} \in \lambda(t_{m+p}, z_{m+p})} (z_{m+p+1} \in A) . \end{cases}$$

So, for $y \in P(A, n)$ the condition being the right hand side of the equivalence (2.4) is satisfied in particular for $t_{m+1} = \ldots = t_{m+p} = 0$. Thus, by virtue of the equality

$$\lambda(0, z_m) = \{z_m\}$$

we get the implication

$$(2.5)\qquad y \in P(A, n) \Rightarrow y \in P(A, m) .$$

By using the same arguments we prove that

$$y \in P(A, 1) \Rightarrow y \in A ,$$

which gives, together with (2.5), the condition (2.3) for $m \leqslant n$. ■

COROLLARY. *For every $A \in \mathscr{P}(X)$ the inclusion*

$$(2.6)\qquad E(A) \subset A$$

holds true.

PROPOSITION 2.2. *If $A \in \mathscr{P}(X)$, then*

$$E(A) = E(E(A)). \tag{2.7}$$

Proof. We get immediately

$$E(E(A)) \subset E(A) \tag{2.8}$$

from (2.6). In order to show the inverse inclusion suppose that $y \in E(A)$. We have to prove that $y \in P(E(A), n)$ for every n. So let n be fixed. We are going to prove that

$$\begin{gathered} \forall_{t_0} \exists_{z_1 \in \lambda(t_0, y)} \forall_{t_1} \exists_{z_2 \in \lambda(t_1, z_1)} \cdots \\ \cdots \forall_{t_n} \exists_{z_{n+1} \in \lambda(t_n, z_n)} (z_{n+1} \in E(A)) \end{gathered} \tag{2.9}$$

The condition

$$z_{n+1} \in E(A) \tag{2.10}$$

is equivalent to the condition

$$z_{n+1} \in P(A, m) \quad \text{for every } m \in N.$$

This means that for every fixed $m \in N$

$$\begin{gathered} \forall_{s_0} \exists_{w_1 \in \lambda(s_0, z_{n+1})} \forall_{s_1} \exists_{w_2 \in \lambda(s_1, w_1)} \cdots \\ \cdots \forall_{s_m} \exists_{w_{m+1} \in \lambda(s_m, w_m)} (w_{m+1} \in A) \end{gathered} \tag{2.11}$$

Comparing (2.9) and (2.11) we observe that

$$y \in P(E(A), n) \quad \text{for every } n \in N \tag{2.12}$$

if and only if

for every $n \in N$ and every $m \in N$ the following condition (2.13) holds true:

$$\begin{gathered} \forall_{t_0} \exists_{z_1 \in \lambda(t_0, y)} \forall_{t_1} \exists_{z_2 \in \lambda(t_1, z_1)} \cdots \\ \cdots \forall_{t_{n+m+1}} \exists_{z_{n+m+2} \in \lambda(t_{n+m+1}, z_{n+m+1})} (z_{n+m+2} \in A). \end{gathered} \tag{2.13}$$

(the dummy variables $s_0, \ldots, s_m$ and $w_1, \ldots, w_{m+1}$ in (2.11) are replaced here by $t_{n+1}, \ldots$ $\ldots, t_{n+m+1}$ and $z_{n+1}, \ldots, z_{n+m+2}$ respectively). The condition (2.13) is however, satisfied since $y \in P(A, n+m)$ for every $n, m \in N$. So (2.12) holds true and then $y \in E(E(A))$. Thus we have proved the inclusion

$$E(A) \subset E(E(A)) \tag{2.14}$$

which in view of (2.8) finishes the proof. ■

Let us establish finally, almost obvious

PROPOSITION 2.3. *If $A, B \in \mathscr{P}(X)$ and $A \subset B$ then*

$$E(A) \subset E(B).$$

Definition 2.3. The set M is said to be $\{\Omega, \beta\}$-stable (shortly: $M \in S\{\Omega, \beta\}$) if and only if

$$\forall_{Q \in \Omega} \forall_{x \in M} \exists_{B \in \beta(x)} (B \subset E(Q)). \tag{2.15}$$

Definition 2.4. We say that M satisfies the condition $L\{\Omega, \beta\}$ (shortly: $M \in L\{\Omega, \beta\}$) if and only if there exists a family $\mathscr{A} \in \mathscr{P}(\mathscr{P}(X))$ such that

$$\beta(y) \succ \mathscr{A} \quad \text{for every } y \in M, \tag{2.16}$$

$$\mathscr{M} \succ \Omega \tag{2.17}$$

$$A \subset E(A) \quad \text{for every } A \in \mathscr{A}. \tag{2.18}$$

($\mathscr{C} \succ \mathscr{D}$ means that for every $D \in \mathscr{D}$ there is $C \in \mathscr{C}$ such that $C \subset D$).

Remark 2.2. The Corollary of Proposition 2.1 permits us to notice that (2.18) is equivalent to

$$A = E(A) \quad \text{for } A \in \mathscr{A}. \tag{2.19}$$

THEOREM 2.1. *M is $\{\Omega, \beta\}$-stable if and only if M satisfies the condition $L\{\Omega, \beta\}$.*

Proof. Assume that $M \in S\{\Omega, \beta\}$. Take

$$\mathscr{A} := \{E(Q) \colon Q \in \Omega\}. \tag{2.20}$$

We claim that $\mathscr{A}$ satisfies the conditions (2.16)—(2.18). First of all notice that $\mathscr{A} \neq \emptyset$, since $\Omega \neq \emptyset$. In order to show (2.16) we observe that for every $y \in M$ and every $A = E(Q)$ we can find $B \in \beta(y)$ such that $B \subset E(Q)$; the set B is chosen for y and Q according to the $\{\Omega, \beta\}$-stability of M. This also proves automatically that every element A of the family $\mathscr{A}$ is non-empty.

The condition (2.17) is obvious because of (2.6): for every $Q \in \Omega$ there is A $(= E(Q))$ belonging to $\mathscr{A}$, such that $A \subset Q$.

Finally we use Proposition 2.2 and we get directly (2.18).

So we have proved that $\{\Omega, \beta\}$-stability of M implies the condition $L\{\Omega, \beta\}$.

Assume now that $M \in L\{\Omega, \beta\}$. Let $x \in M$ and $Q \in \Omega$ be given. Applying (2.17) we choose $A \in \mathscr{A}$ such that $A \subset Q$. Now, by virtue of (2.16) we can find $B \in \beta(x)$ such that $B \subset A$, and so $B \subset E(Q)$ since $A \subset E(A) \subset E(Q)$. The condition (2.15) is then satisfied. Thus the condition $L\{\Omega, \beta\}$ implies the condition: $M \in S\{\Omega, \beta\}$. ■

PROPOSITION 2.4. *The condition at the right hand side of the equivalence* (2.1) *defining $P_\lambda(A, n)$ is equivalent to the following condition*

$$\begin{aligned} &\forall_{t_0 \in G} \exists_{z_1 \in \lambda(t_0, y) \cap A} \forall_{t_1 \in G} \exists_{z_2 \in \lambda(t_1, z_1) \cap A} \cdots \\ &\cdots \forall_{t_{n-1} \in G} \exists_{z_n \in \lambda(t_{n-1}, z_{n-1}) \cap A} \forall_{t_n \in G} \quad (\lambda(t_n, z_n) \cap A \neq \emptyset) \end{aligned} \tag{2.21}$$

Proof. The implication (2.21) $\Rightarrow$ (2.1) is trivial. In order to show that (2.1) implies (2.21) observe that if for $z_1 \in \lambda(t_0, y)$ certain condition should be satisfied for every t_1;

then in particular this condition has to be satisfied for $t_1 = 0$. But the existence of z_2 belonging to $\lambda(0, z_1) \cap A$ means that $\emptyset \neq \lambda(0, z_1) \cap A = \{z_1\} \cap A$ and so $z_1 \in A$. The same arguments applied with respect to $z_2, \ldots, z_n$ permit us to conclude that in (2.1) $z_i \in \lambda(t_{i-1}, z_{i-1}) \cap A$. Thus (2.1) $\Rightarrow$ (2.21). ■

Remark 2.3. The condition (2.1) is essentially stronger than

$$\forall_t\colon \lambda(t, y) \cap A \neq \emptyset \tag{2.22}$$

which is (in the case of a regular system) equivalent to the following one

$$\forall_n \forall_{(t_0, \ldots, t_n) \in G^n} \exists_{(z_1, \ldots, z_{n+1}) \in X^n} (z_{i+1} \in \lambda(t_i, z_i),\ z_{n+1} \in A) \tag{2.23}$$

(with $z_0 := y$).

Remark 2.4. If for every (t, x) the set $\lambda(t, x)$ consists of a single element, then instead of the generalized system $(X, G; \lambda)$ we can consider a system $(X, G; \pi)$ with $\pi(t, x)$ defined in such a way that $\{\pi(t, x)\} = \lambda(t, x)$.

In that case we can easily show that the condition (2.1) is equivalent to

$$\pi(t, y) \in A \quad \text{for every } t \in G \tag{2.24}$$

and so, the $\{\Omega, \beta\}$-stability is in fact equivalent to the Lyapunov type stability considered in [3] and [4]:

$$\forall_{x \in M} \forall_{Q \in \Omega} \exists_{B \in \beta(x)} (\pi(B) \subset Q) \tag{2.25}$$

where $\pi(B) = \{\pi(t, x)\colon\ t \in G, x \in B\}$; this is true because of the equivalence: $\pi(B) \subset Q \Leftrightarrow B \subset E(Q)$, where — in our case — $E(Q)$ can be defined as the set of these $y \in X$ for which the trajectory $\pi(y) = \{\pi(t, y)\colon\ t \in G\}$ is contained in Q.

For such stability conditions Lyapunov functions are discussed in [3], [4].

Conditions of the type (2.25) with respect to "multivalued" generalized systems $(X, G; \lambda)$ were considered in [5]. It is clear that the condition

$$\forall_{x \in M} \forall_{Q \in \Omega} \exists_{B \in \beta(x)} \big(\lambda(B) = \bigcup \{\lambda(t, x)\colon\ t \in G, x \in B\} \subset Q\big)$$

is essentially stronger than (2.15).

3. Let $(X, G; \lambda)$ be a generalized system M, Ω and β be as in Sec. 2. Suppose that $N \in \mathcal{P}(X)$ is such that

$$N \subset E(N). \tag{3.1}$$

Assume that $(T, \leqslant)$ is a partially ordered space and put

$$P := T \setminus \{\inf T\} \quad \text{if } \inf T \text{ exists}$$

and

$$P = T \quad \text{if } \inf T \text{ does not exist}.$$

Definition 3.1. A function $V\colon N \to T$ is said to be a Lyapunov function of the type $\{N, T; \Omega, \beta\}$ for the set M if and only if the family

$$\mathcal{A} = \{A_\eta\colon\ \eta \in P\} \tag{3.2}$$

where

(3.3) $$A_\eta := \{x \in N:\ V(x) < \eta\} \quad \text{for } \eta \in P$$

satisfies the conditions (2.16) and (2.17) and moreover

(3.4) $$\forall_{x \in N} \forall_{\mu > V(x)} \big(x \in E(A_\mu)\big).$$

THEOREM 3.1. *If there exists a Lyapunov function of the type $\{N, T;\ \Omega, \beta\}$ for M, then $M \in S\{\Omega, \beta\}$.*

Proof. We apply Theorem 2.1. It is explicitely assumed that $\mathscr{A}$ given by (3.2)—(3.3) satisfies (2.16)—(2.17). The condition (2.18) is also satisfied since (3.4) gives:

$$x \in A_\eta \Rightarrow x \in N,\ V(x) < \eta \Rightarrow x \in E(A_\eta). \ \blacksquare$$

The above Theorem 3.1 says that the existence of a Lyapunov function is a sufficient condition for M to be $\{\Omega, \beta\}$-stable. We shall prove that this condition is also necessary for the $\{\Omega, \beta\}$-stability, if T satisfies some additional conditions and Ω is indexed by P monotonically.

First of all we introduce the following

Definition 3.2. We say that the partially ordered space $(T, \leqslant)$ satisfies the condition:

$$(C_1) \overset{\text{df}}{\Leftrightarrow} \text{for every } S \in \mathscr{P}(T) \text{ there exists } \inf S \in T;$$

$$(C_2) \overset{\text{df}}{\Leftrightarrow} \text{for every } \eta \in P \text{ there exists } \mu \in P \text{ such that } \mu < \eta;$$

$$(C_3) \overset{\text{df}}{\Leftrightarrow} \text{for every } S \in \mathscr{P}(T) \text{ and every } \eta \in P \text{ such that}$$
$$\inf S < \eta, \text{ there is } \sigma \in S \text{ such that } \sigma \leqslant \eta,$$

(see [3], [4]).

LEMMA 3.1. (see [4]). *The conditions (C_1)—(C_3) for $(T, \leqslant)$ imply the following condition*

(C_4) $$\text{if } \eta, \tilde{\eta} \in P \text{ then } \inf\{\eta, \tilde{\eta}\} \in P.$$

Proof. First of all observe that for $\eta, \tilde{\eta} \in P$ the element $\mu := \inf\{\eta, \tilde{\eta}\}$ exists because of the condition (C_1). Suppose that μ does not belong to P; so $\mu = \inf T$. Applying (C_2) we can find $\sigma \in P$ such that $\sigma < \eta$. Of course

(3.5) $$\mu = \inf T < \sigma.$$

So, by virtue of (C_3), there exists $\varrho \in \{\eta, \tilde{\eta}\}$ such that $\varrho \leqslant \sigma$. Because of the strict inequality $\sigma < \eta$, we get: $\varrho < \eta$. Thus ϱ must be equal to $\tilde{\eta}$. Thus $\tilde{\eta} \leqslant \sigma$ and consequently $\tilde{\eta} < \eta$ and $\tilde{\eta} = \min\{\eta, \tilde{\eta}\} = \inf\{\eta, \tilde{\eta}\} = \mu$. The last equality contradicts (3.5). The proof is completed. $\blacksquare$

THEOREM 3.2. *Assume that $(T, \leqslant)$ satisfies the conditions (C_1)—(C_3). Suppose that*

$$\Omega = \{Q_\eta \colon \eta \in P\}$$

is such that

$$\eta < \mu \Rightarrow Q_\eta \subset Q_\mu .$$

If $M \in S\{\Omega, \beta\}$, then there exists a non-empty subset N of X satisfying (3.1) *and such that*

$$\forall_{x \in N} \exists_{B \in \beta(x)} (B \subset N) \tag{3.6}$$

and there exists a Lyapunov function $V\colon N \to T$ of the type $\{N, T;\ \Omega, \beta\}$ for the set M.

Proof. I. Let $\eta^0 \in P$ be fixed. We put

$$N := E(Q_{\eta 0}) . \tag{3.7}$$

We have $N = E(N)$ (see Proposition 2.2) and so (3.1) is trivially satisfied. Moreover, if $x \in M$, then there exists $B \in \beta(x)$ such that $B \subset E(Q_{\eta 0})$ (because of the $\{\Omega, \beta\}$-stability of M). Thus (3.6) holds true for N defined by (3.7).

II. We define V by the formula

$$V(x) := \inf\{\eta \in P\colon\ x \in E(Q_\eta)\} , \qquad x \in N . \tag{3.8}$$

We shall show that the family $\mathscr{A}$ given by (3.2)—(3.3) for the function (3.8) above satisfies (2.16) and (2.17).

Let $x \in M$ and $A_\eta \in \mathscr{A}$ be given. Put

$$\mu := \inf\{\eta, \eta^0\} . \tag{3.9}$$

Notice that $\mu \in P$ (see Lemma 3.1) and then there exists $\sigma \in P$ such that $\sigma < \mu$ (see (C_2)). Let us consider Q_σ. For x and Q_σ we can find $B \in \beta(x)$ such that $B \subset E(Q_\sigma)$. Since $Q_\sigma \subset Q_\mu \subset Q_{\eta 0}$, we have (see Proposition 2.3)

$$E(Q_\sigma) \subset E(Q_\mu) \subset E(Q_{\eta 0}) = N$$

and so $B \subset N$. Thus for every $z \in B$ the value $V(z)$ is well defined. Moreover

$$z \in B \Rightarrow z \in E(Q_\sigma) \Rightarrow V(z) \leqslant \sigma \Rightarrow V(z) < \mu \Rightarrow V(z) < \eta$$

and so $B \subset A_\eta$. We have proved that $\mathscr{A}$ satisfies (2.16).

In order to prove (2.17) consider any fixed $Q_\eta \in \Omega$. Take $\mu \in P$ such that $\mu < \eta$ (compare (C_2)). Take now $y \in A_\mu$. We have

$$V(y) < \mu$$

and so

$$\inf\{\sigma \in P\colon\ y \in E(Q_\sigma)\} < \mu$$

Because of (C_3) we can find $\sigma \in P$ such that $\sigma \leqslant \mu$ and $y \in E(Q_\sigma)$. We have

$$E(Q_\sigma) \subset Q_\sigma \subset Q_\mu \subset Q_\eta$$

and then $y \in Q_\eta$. Thus we have proved that

$$A_\mu \subset Q_\eta \tag{3.10}$$

Our result is: for every $Q_\eta \in \Omega$ there is $A_\mu \in \mathscr{A}$ (it is enough to take any $\mu \in P$ such that $\mu < \eta$), such that (3.10) holds true. This means that the condition (2.17) is fulfilled.

III. We are going to complete the proof by proving that the function V defined by (3.8) satisfies the condition (3.4). Let $x \in N$ and $\mu \in P$ such that $V(x) < \mu$ be given. Take $\eta \leqslant \mu$ such that $x \in E(Q_\eta)$; we get

$$E(Q_\eta) \subset E(Q_\mu)$$

and so $x \in E(A_\mu)$. ■

4. Let $(X, G; \lambda)$ be a generalized system and let $M \in \mathscr{P}(X)$, Ω, β be as in Sec. 2

PROPOSITION 4.1. *Assume that* $N \in \mathscr{P}(X)$ *satisfies* (3.1). *Let* V *be a function defined in* N, *ranged in* T *where* $(T, \leqslant)$ *is a dense ordered space satisfying the conditions* (C_1) *and* (C_3) *(see Definition* 3.2). *Then the family* $\mathscr{A}$ *defined by* (3.2)—(3.3) *satisfies the conditions* (2.16)—(2.17) *if and only if these two conditions are fulfilled by the family*

$$\bar{\mathscr{A}} = \{\bar{A}_\eta\},$$

where

$$\bar{A}_\eta := \{x \in N\colon\ V(x) \leqslant \eta\}. \tag{4.1}$$

Moreover, the condition (3.4) *is satisfied if and only if*

$$\forall_{x \in N} \forall_{\mu > V(x)} (x \in E(\bar{A}_\mu)) \tag{4.2}$$

Proof. In order to prove the first statement it is enough to observe that

$$A_\eta \subset \bar{A}_\eta \quad \text{for } \eta \in P \tag{4.3}$$

and simultaneously

$$\bar{A}_\eta \subset A_\mu \quad \text{for } \eta < \mu. \tag{4.4}$$

Proving the second statement, saying that (3.4) is equivalent to (4.2), we notice first that the implication: (3.4) $\Rightarrow$ (4.2) is obvious, since $A_\eta \subset \bar{A}_\eta$ and so

$$x \in E(A_\eta) \Rightarrow x \in E(\bar{A}_\eta) \quad \text{for } \eta \in P.$$

Then we prove the inverse implication: (4.2) $\Rightarrow$ (3.4) as follows. Take $x \in N$ and $\mu > V(x)$ and then fix $\eta \in P$ such that $V(x) < \eta < \mu$. (This is possible because of the density assumed for the space T). We have: $x \in E(\bar{A}_\eta)$ and so $x \in E(A_\mu)$ by virtue of (4.4). ■

COROLLARY. *If* $(T, \leqslant)$ *is ordered and dense then in the formulation of Definition* 3.1 *we can replace the family* $\mathscr{A}$ *by* $\bar{\mathscr{A}}$ *considered above.*

Remark 4.1. Recall that in Sec. 3 T is assumed to be partially ordered. It seems to be natural to ask whether the assumption that T is ordered is essential.

The answer is given by the