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Multiple positive solutions for a class of nonlinear boundary value problems of the fourth order

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Introduction. In this paper we investigate the existence and multiplicity of positive solutions for a class of nonlinear boundary value problems of the fourth order. The results of this paper generalize the results of paper [4]. Namely we transfer some results of paper [4] relative to differential equation of the second order to some class of differential equation of the fourth order. The method used in this paper is based principally on the results of papers [2], [3] and [4].

Consider the semilinear elliptic eigenvalue problem

$$(1)\qquad \begin{aligned} (Lu)(x) &= \lambda f(x, u(x)) && \text{for } x \in \Omega, \\ (Bu)(x) &= 0 && \text{for } x \in \partial\Omega, \end{aligned}$$

where $L = L_1 L_0$, $B = (B_0, B_1)$, $\lambda > 0$.

We assume that $\Omega \subset R^n$ is a bounded domain with boundary $\partial\Omega$ of class $C^{2+\alpha}$ for some $\alpha \in (0, 1)$, that differential operators

$$(2)\qquad L_i \varphi = -\sum_{k,j=1}^{n} a^i_{jk} \partial_k \partial_j \varphi + \sum_{k=1}^{n} a^i_k \partial_k \varphi + c^i \varphi, \qquad i = 0, 1$$

are uniformly elliptic on Ω and that the coefficients a^i_{kj}, a^i_k, $c^i \in C^{2-2i+\alpha}(\Omega)$ $(i = 0, 1)$ with c^i $(i = 0, 1)$ nonnegative. We also require that

$$(3)\qquad (B_i \varphi)(x) := b^i_0(x) \varphi(x) + b^i_1(x) \frac{\partial \varphi}{\partial \beta}(x), \qquad i = 0, 1$$

and $(Bu)(x) = 0$ on $\partial\Omega$ means $(B_0 u)(x) = 0$ and $(B_1 L_0 u)(x) = 0$ for $x \in \partial\Omega$, where $\partial/\partial\beta$ denote the directional derivative with respect to an outward pointing, nowhere tangent vector field β on $\partial\Omega$ of class $C^{1+\alpha}$ and that $b^i_0 \equiv 1$ and $b^i_1 \equiv 0$ (Dirichlet boundary condition) or that $b^i_1 \equiv 1$ and $b^i_0 \in C^{1+\alpha}(\partial\Omega)$ $(i = 0, 1)$ with $b^i_0 \geqslant 0$ but not identically zero. These hypotheses ensure that the maximum principle is valid for (L_i, B_i) on Ω, $i = 0, 1$.

We make the following assumptions on f (see [4])

(f_1) $f \in C^1(\bar{\Omega}x[0, +\infty))$
(f_2) $f(x,0) > 0$ for $x \in \Omega$
(f_3) There exists $r > 0$ such that $f(x,s) > 0$ if $s \in (0,r)$ and $f(x,r) \leqslant 0$ for $x \in \Omega$.

We prove in Section 1 that the above hypotheses imply that problem (1) possesses a minimal positive solution for all $\lambda > 0$. Our proof makes use of upper and lower solutions. We shall say that u is an upper solution of problem (1) if

$$(Lu)(x) \geqslant \lambda f(x, u(x)) \quad \text{for } x \in \Omega$$

and

$$(Bu)(x) \geqslant 0 \quad \text{for } x \in \partial\Omega,$$

where $(Bu)(x) \geqslant 0$ for $x \in \partial\Omega$ means $(B_0 u)(x) \geqslant 0$ and $(B_1 L_0 u)(x) \geqslant 0$ for $x \in \partial\Omega$. Lower solutions are defined similary but with the inequality signs reversed.

The main result in Section 1 (Theorem 1.1) states that an a priori bound on the upper solutions of problem (1). In Section 2 we show that such a priori bounds can be found in superlinear ordinary differential equations.

1. Existence of positive solutions of problem (1)

First we recall well known result from the theory linear differential equations elliptic type of second order (e.g., see [1] or [4]).

For every $v \in C^\alpha(\bar{\Omega})$ the linear boundary value problem

$$(4) \qquad \begin{aligned} L_i u &= v \quad \text{in } \Omega \\ B_i u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

has a unique solution $u = K_i v \in C^{2+\alpha}(\bar{\Omega})$ for $i = 0, 1$. The Schauder a priori estimates imply that these inverse mappings K_i ($i = 0, 1$) are continuous linear operators from $C^\alpha(\bar{\Omega})$ to $C^{2+\alpha}(\bar{\Omega})$. Each operators K_i, $i = 0, 1$ can be extended to a compact linear operator from $C(\bar{\Omega})$ to $C^1(\bar{\Omega})$.

Let us choose and fix $\lambda > 0$. Using the definition of the operator K_1 we write the problem (1) in the form

$$(5) \qquad \begin{aligned} (L_0 u)(x) &= \lambda (K_1 f)(x, u(x)) \quad &&\text{for } x \in \Omega \\ (B_0 u)(x) &= 0 &&\text{for } x \in \partial\Omega. \end{aligned}$$

Since $f \in C^1(\bar{\Omega}x[0, +\infty))$ and $K_1\colon C(\bar{\Omega}) \to C^1(\bar{\Omega})$, there exist $m > 0$ such that $\frac{\partial}{\partial u}(K_1 f)(x, u) > -m$ for all $x \in \Omega$ and $u \in [0, r]$.

Now we consider the problems

$$(6) \qquad \begin{aligned} L_m u &= (L_0 + \lambda m) u = v \text{ in } \Omega \\ B_0 u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

and

$$(7)\qquad \begin{aligned} &(L_m u)(x) = \lambda[(K_1 f)(x, u(x)) + mu(x)] = g(x, u(x)) \quad \text{for } x \in \Omega \\ &(B_0 u)(x) = 0 \quad \text{for } x \in \partial\Omega . \end{aligned}$$

Because of the definition of m, $g(x, \cdot)$ is positive and strictly increasing for $u \in [0, r]$. We can extend g to $\bar{\Omega}\, x(-\infty, +\infty)$ so that $g \in C^1(\bar{\Omega}x(-\infty, +\infty))$ and $g(x, \cdot)$ is positive and strictly increasing for $u \in (-\infty, r]$ (see [4]).

Let K_m denote the inverse mapping corresponding to (6). Then K_m: $C(\bar{\Omega}) \to C^1(\bar{\Omega})$ is compact. Since g is continuous, the Nemytskii operator N: $C(\bar{\Omega}) \to C(\bar{\Omega})$ such that $(Nu)(x) := g(x, u(x))$ is continuous. Let us denote by $T = K_m N$, then T: $C(\bar{\Omega}) \to C(\bar{\Omega})$ is compact and continuous.

Analogously as in the paper [4] it is easy to show the following lemmas

LEMMA 1.1. *If $u \in C(\bar{\Omega})$ and $u(x) \leqslant r$ for $x \in \Omega$, then $(Tu)(x) > 0$ for $x \in \Omega$ and $\partial/\partial\beta(Tu)(x) \leqslant 0$ for $x \in \partial\Omega$.*

LEMMA 1.2. *If $u, v \in C(\bar{\Omega})$, $u(x) \leqslant v(x) \leqslant r$ for $x \in \Omega$ and $u \neq v$ then $(Tu)(x) \leqslant (Tv)(x)$ for $x \in \Omega$.*

LEMMA 1.3. *If $v_0 = r$, then $(Tv_0)(x) < r$ for $x \in \bar{\Omega}$.*

The proofs of Lemmas 1.1, 1.2 and 1.3 are simple consequence of the maximum principle for (L_m, B_0) in Ω and the fact that the function Tu satisfies the following problem

$$(8)\qquad \begin{aligned} &(L_m Tu)(x) = g(x, u(x)) \quad \text{for } x \in \Omega \\ &(B_0 Tu)(x) = 0 \qquad\quad\ \text{for } x \in \partial\Omega \end{aligned}$$

(see [4]).

In the sequel our main tool is the maximum principle for L_i on Ω ($i = 0, 1$), which we use in the following form (see [5] or [4]).

THEOREM (Maximum principle). *Let u satisfy the differential inequality*

$$(L_i u)(x) \geqslant 0 \quad \text{for } x \in \Omega,$$

where $i = 0$ or $i = 1$. If u attains a nonpositive minimum M at $x \in \Omega$, then $u \equiv M$. Also, if u attains a nonpositive minimum M at a point $x_0 \in \partial\Omega$ and if $u \not\equiv M$ on Ω, then $(\partial u/\partial\beta)(x_0) < 0$, where $\partial/\partial\beta$ is any outward pointing directional derivative.

Now using a degree theory argument and above lemmas we can prove the existence of a positive solution of problem (1). Let us denote by $B_r := \{u \in C(\bar{\Omega})\colon \|u\| < r\}$ the ball in $C(\bar{\Omega})$. (Here $\|\cdot\|$ denotes the usual sup norm for $C(\bar{\Omega})$).

We have the following

THEOREM 1.1. *For each fixed $\lambda > 0$ 1° there exists a positive solution $u \in B_r$ of problem* (1). *2° If there exists $K > 0$ such that $\|u\| < K$ for all upper solutions of problem* (7) *satisfying $(B_0 u)(x) = 0$ for $x \in \partial\Omega$, then there exist at least two positive solutions of problem* (1).

The proof of Theorem 1.1 is omitted because it is almost identical to the proofs of theorems 2.4 and 2.5 of paper [4].

Remark 1.1. Let us observe that each upper solution of problem (1) satisfying $(Bu)(x) = 0$ for $x \in \partial\Omega$ is an upper solution of problem (7) satisfying $(B_0 u)(x) = 0$ for $x \in \partial\Omega$, but not conversely.

Remark 1.2. Evidently, each solution of problem (1) is a solution of problem (7) and vice versa.

We now investigate the minimal positive solution of problem (1). By the condition (f_2), function $u = 0$ in Ω is a lower solution of the problem (1) (and so of the problem (7)). By (f_3), $u = r$ in Ω is an upper solution of problem (7). It is proved by Sattinger in [5] that the iterations defined by

$$u_0 = 0 \text{ and } u_{n+1} = Tu_n;\ v_0 = r \text{ and } v_{n+1} = Tv_n,\ n \in N$$

converge uniformly in $\bar{\Omega}$. Using the method of Sattinger from paper [5] we may prove that $\{u_n\}$ converges uniformly to $u \in C^{4+\alpha}(\bar{\Omega})$ and that u is a minimal positive solution of problem (7). By Remark 1.2 the function u defined above is a solution of problem (1). We shall prove that u is a minimal solution of problem (1). Indeed if there exists $\bar{u} \in C^{4+\alpha}(\bar{\Omega})$ such that $\bar{u}(x) < u(x)$ for $x \in \Omega$ and $\bar{u}$ is the solution of problem (1), then $\bar{u}$ is a solution of problem (7) and $\bar{u}(x) < u(x)$ for $x \in \Omega$. This is a contradiction because u is a minimal solution of problem (7).

2. A priori bounds for upper solutions

In this section we obtain a priori bounds on the upper solutions of certain superlinear fourth order problems of the Sturm–Liouville type. We use an argument almost identical with that used by Brown and Budin in [4] to obtain a priori bounds for Sturm-Liouville problems of second order. We consider the following problem

$$(9)\qquad \begin{aligned} &(Lu)(t) = \lambda f(t, u(t)) \quad \text{for } t \in [0, 1] \\ &(Bu)(t)|_0^1 = 0, \end{aligned}$$

where $L = L_1 L_0$, $B = (B_0, B_1)$, $\lambda > 0$, and

$$(9')\qquad (L_i u)(t) = -[p_i(t)u'(t)]' + q_i(t)u(t), \quad i = 0, 1$$

$p_i(t) > 0$, $q_i(t) \geqslant 0$ for $t \in [0, 1]$, $p_i \in C^3([0, 1])$, $q_i \in C^2([0, 1])$ $(i = 0, 1)$. The boundary condition has the form $(Bu)(t)|_0^1 = 0$ i.e., $(B_0 u)(t)|_0^1 = 0$ and $(B_1 L_0 u)(t)|_0^1 = 0$, where $(B_i \varphi)(t)|_0^1 = 0$ means that

$$\varphi(0) - \alpha_i \varphi'(0) = 0 \text{ and } \varphi(1) + \beta_i \varphi'(1) = 0$$
$$\alpha_i \geqslant 0 \text{ and } \beta_i \geqslant 0, \quad (i = 0, 1).$$

If the function f in problem (9) satisfies the conditions (f_1), (f_2) and (f_3), then the results of section 1 hold for the problem (9). In this section we make the following additional assumption on f (see [4])

(f_4) There exist t_0, t_1, with $0 \leqslant t_0 < t_1 \leqslant 1$ such that $\lim_{u \to +\infty} f(t, u)/u = +\infty$ uniformly in $t \in [t_0, t_1]$.

We shall prove the following

THEOREM 2.1. *Let f satisfies the conditions (f_1), (f_2), (f_3) and (f_4). For every $\lambda > 0$ there exists $\varrho > 0$ such that if u is an upper solution of (9) then $\|u\| < \varrho$.*

Proof. Let s_0, s_1 be such that $t_0 < s_0 < s_1 < t_1$. We prove that there exists $\alpha > 0$ such that for every upper solution u of (9).

(10) $$u(t) \geqslant \alpha \|u\| \quad \text{for } t \in [s_0, s_1].$$

For each $s \in [0, 1]$, let $y(\cdot, s)$ be the solution of the boundary value problem

$$(L_m y)(t) = 0 \quad \text{for } t \in (0, s)$$
$$y(0) = 0 \quad \text{and } y(s) = 1.$$

Similarly, for each $s \in [0, 1)$, let $z(\cdot. s)$ be the solution of the boundary value problem

$$(L_m z)(t) = 0 \quad \text{for } t \in (s, 1)$$
$$z(s) = 1 \quad \text{and } z(1) = 0.$$

where $L_m = L_0 + \lambda m$, see (6).

Now we define w: $[0, 1] x [0, 1] \to R$ by

(11) $$w(t, s) = \begin{cases} z(t, 0) & \text{for } 0 \leqslant t \leqslant 1 \text{ and } s = 0 \\ y(t, s) & \text{for } 0 \leqslant t \leqslant s \text{ and } 0 < s < 1 \\ z(t, s) & \text{for } s \leqslant t \leqslant 1 \text{ and } 0 < s < 1 \\ y(t, 1) & \text{for } 0 \leqslant t \leqslant 1 \text{ and } s = 1. \end{cases}$$

Clearly, w is continuous and strictly positive on $[s_0, s_1] x\ [0, 1]$. Hence

$$\alpha = \min\{w(t, s) \colon (t, s) \in [s_0, s_1] x\ [0, 1]\} > 0.$$

Let u be an arbitrary upper solution of problem (9). Then there exists $\sigma_0 \in [0, 1]$ such that $u(\sigma_0) = \|u\|$. We prove that

$$u(t) \geqslant \|u\| w(t, \sigma_0) \quad \text{for } t \in [0, 1].$$

Let $M_i([a, b])$, $i = 0, 1$ stand for the set of functions φ: $[a, b] \to R$ of class C^2 in $[a, b]$, which satisfy the following boundary conditions

$$\varphi(a) - \alpha_i \varphi'(a) = 0 \text{ and } \varphi(b) + \beta_i \varphi'(b) = 0 \quad \text{for } i = 0, 1$$

where $\alpha_i \geqslant 0$ and $\beta_i \geqslant 0$ $(i = 0, 1)$ and $a, b \in R$.
Operators

$$L_i \colon M_i([a, b_i]) \to C([a, b]), \quad i = 0, 1$$

where L_i, $i = 0, 1$ are defined by the formula (9′), are linear and inversible operators of the Sturm-Liouville type.

Let us denote by K_1 the inverse operator to the operator L_1 when $[a, b] = [0, 1]$. Now, the problem (9) may be written in the form

$$(12) \qquad \begin{aligned} &(L_m u)(t) = g(t, u(t)) \quad \text{for } t \in [0, 1] \\ &u(0) - \alpha_0 u'(0) = 0 \text{ and } u(1) + \beta_0 u'(1) = 0, \end{aligned}$$

where $g(t, u) := \lambda(K_1 f)(t, u) + \lambda m u$.

By the definition of function w, we have

$$L_m(u - \|u\| w(\cdot, \sigma_0))(t) = (L_m u)(t) \geqslant g(t, u(t)) \quad \text{for } 0 < t < \sigma_0 .$$

We observe also that $u(0) - \|u\| w(0, \sigma_0) = u(0) \geqslant 0$ and $u(\sigma_0) - \|u\| w(\sigma_0, \sigma_0) = \|u\| - \|u\| = 0$. Because $g(t, u(t)) \geqslant 0$ for $t \in [0, 1]$ hence, by the maximum principle, $u(t) \geqslant \|u\| w(t, \sigma_0)$ for $t \in (0, \sigma_0)$. A similar argument shows that $u(t) \geqslant \|u\| w(t, \sigma_0)$ for $t \in (\sigma_0, 1)$. Since $u(\sigma_0) = \|u\| = \|u\| w(\sigma_0, \sigma_0)$ hence $u(t) \geqslant \|u\| w(t, \sigma_0)$ for $t \in [0, 1]$. Consequently, if $t \in [s_0, s_1]$, then $u(t) \geqslant \alpha \|u\|$.

Let μ_1 be the first eigenvalue of the following problem

$$(13) \qquad \begin{aligned} &(L_0 L_1 y)(t) = \mu y(t) \quad \text{for } s_0 < t < s_1 \\ &y(s_0) = y(s_1) = 0 \text{ and } (L_1 y)(s_0) = (L_1 y)(s_1) = 0 . \end{aligned}$$

As we know $\mu_1 > 0$ and the first eigenfunction Φ of problem (13) is positive in (s_0, s_1) (see [6]). From the condition (f_4) follows that there exists $k > 0$ such that $\lambda f(t, u) > \mu_1 u$ for $t \in [s_0, s_1]$ and $u > k$ ($\lambda > 0$ is fixed). Suppose that u is an upper solution of (9) with $\|u\| > \frac{k}{\alpha}$. From this by (10) we get $u(t) > k$ for $t \in [s_0, s_1]$, and so

$$(L_1 L_0 u)(t) = \lambda f(t, u(t)) > \mu_1 u(t) \quad \text{for } t \in (s_0, s_1) .$$

Since

$$(L_0 L_1 \Phi)(t) = \mu_1 \Phi(t) \quad \text{for } t \in (s_0, s_1)$$

we have

$$(14) \qquad \Phi(t)(L_1 L_0 u)(t) - u(t)(L_0 L_1 \Phi)(t) > 0 \quad \text{for } t \in (s_0, s_1) .$$

On the other hand, using the integration-by-part formula and using the boundary condition $\Phi(s_0) = \Phi(s_1) = (L_1 \Phi)(s_0) = (L_1 \Phi)(s_1) = 0$, we obtain

$$(15) \qquad \int_{s_0}^{s_1} [\Phi(L_1 L_0 u) - u(L_0 L_1 \Phi)](t)\, dt = [p_1(t) \Phi'(t)(L_0 u)(t) + p_0(t) u(t)(L_1 \Phi)'(t)]\Big|_{s_0}^{s_1} .$$

Since $p_1(t) > 0$, $p_0(t) > 0$, $(L_0 u)(t) > 0$ and $u(t) > 0$ for $t \in [s_0, s_1]$ while $\Phi'(s_0) > 0$, $\Phi'(s_1) < 0$ and $(L_1 \Phi)'(s_0) > 0$, $(L_1 \Phi)'(s_1) < 0$, the right side of (15) is negative. Hence we get a contradiction with (14). So every upper solution of problem (9) satisfies the inequality $\|u\| \leqslant \frac{k}{\alpha}$. The proof of Theorem 2.1 is complete.

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