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On a maximum principle for fourth order ordinary differential inequalities

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1. Introduction. It is well known that for the equations and inequalities of second order the so called maximum principle holds. It is also known that maximum principle in general is not true for the equations and inequalities of higher order even for the ordinary equations and inequalities of the fourth order (see [2]).

However in paper [1], the following result was proved. Let u be a real-valued function of class C^4 on the interval $[a, b]$. Suppose u satisfies the inequalities

$$u^{(4)}(x) \geqslant 0, \qquad x \in (a, b)$$
$$u'(a) \geqslant 0, \qquad u'(b) \leqslant 0,$$

and moreover attains its minimum at a point $x_0 \in (a, b)$. Then u is constant on $[a, b]$.

Next this result was generalized in paper [2], where among other is proved the following

Theorem 1. *Let* $u \in C^4(a, b) \cap C^2[a, b]$ *satisfy the differential inequalities*

$$u^{(4)}(x) + g(x)u'''(x) + h(x)u''(x) \geqslant 0, \; x \in (a, b) \tag{1.1}$$
$$u'(a) \geqslant 0, \; u'(b) \leqslant 0,$$

where the functions g *and* h *are bounded on every closed subinterval of* (a, b). *If there exists a function* $w \in C^2[a, b]$ *such that*

$$w(x) > 0, \qquad x \in [a, b]$$
$$w''(x) + g(x)w'(x) + h(x)w(x) \leqslant 0, \qquad x \in (a, b),$$

then u *cannot assume a minimum value at an interior point of* (a, b) *unless* u *is identically constant.*

As well in paper [2] (see Remark 3) the author observes that Theorem 1 is false if the inequality (1.1) contains a component with the unknown function u or its first derivative.

In this paper we shall give further generalizations of Theorem 1 from paper [2] to the case of more general fourth order differential inequalities. In the sequel, analogously as in [2] we shall use the following

THEOREM A (see [3] or [2]). *Let* $v \in C^2(a,b) \cap C[a,b]$ *satisfy the differential inequality*

$$v''(x)+g(x)v'(x)+h(x)v(x) \geqslant 0, \qquad x \in (a,b),$$

where the given functions g and h are bounded on every closed subinterval of (a,b). *Suppose there exists a function* $w \in C^2[a,b]$ *such that*

$$w(x) > 0, \qquad x \in [a,b]$$
$$w''(x)+g(x)w'(x)+h(x)w(x) \leqslant 0, \qquad x \in (a,b).$$

Then v/w *cannot attain a nonnegative maximum in* (a,b) *unless it is constant.*

Remark. The thesis of Theorem A contains second part but we shall not used it in this paper.

2. Maximum principle

Let $\mathfrak{W}$ be the set of functions of class $C^4(a,b) \cap C^2[a,b]$. Consider in $\mathfrak{W}$ a differential operator of the fourth order of the form

$$L = L_1 L_2, \tag{2.1}$$

where L_1 and L_2 denote the following differential operators of the second order

$$L_1\varphi = \varphi''+p_1\varphi'+q_1\varphi \text{ and } L_2\varphi = r\varphi''+p_2\varphi'+q_2\varphi, \ \varphi \in C^2(a,b). \tag{2.2}$$

We assume p_k, q_k $(k = 1, 2)$ and r are given functions such that $p_2, q_2, r \in C^2[a,b]$, $q_2(x) \leqslant 0$, $r(x) > 0$ for $x \in [a,b]$ and p_1, q_1 are bounded on every closed subinterval of (a,b). We also assume that there exists a function $w \in C^2[a,b]$ such that

$$\begin{gathered} w(x) > 0, \qquad x \in [a,b] \\ w''(x)+p_1(x)w'(x)+q_1(x)w(x) \leqslant 0, \qquad x \in (a,b). \end{gathered} \tag{2.3}$$

THEOREM 2.1. *Let* $u \in \mathfrak{W}$ *satisfy the differential inequalities*

$$\begin{gathered} (Lu)(x) \geqslant 0, \qquad x \in (a,b) \\ u(a) \geqslant 0,\ u(b) \geqslant 0 \text{ and } u'(a) \geqslant 0,\ u'(b) \leqslant 0, \end{gathered} \tag{2.4}$$

where L is defined by (2.1) *and the given functions* p_k, q_k $(k = 1, 2)$ *and r satisfy the above assumptions. Then u cannot assume a nonpositive minimum value at an interior point of* (a,b) *unless u is identically zero.*

Proof. The proof of this theorem is similar to the proof of Theorem 1 (see [2]). Suppose u assumes a nonpositive minimum value m at a point of (a, b). Consider the set M of all nonpositive minimum points $x \in (a, b)$, i.e.

$$M = \{x \in (a, b)\colon u(x) = m\}.$$

By hypothesis, M is nonempty and by continuity of u, M is closed relative to (a, b). Since (a, b) is connected, it suffices to show that M is open relative to (a, b), for then, $M = (a, b)$ and by continuity $u = m$ in $[a, b]$. Thus suppose $x_0 \in M$. In view of the boundary conditions at $x = a$, and by $u(x_0) \leqslant 0$, there exists a point $\xi \in [a, x_0)$ such that $u(\xi) = \sup\limits_{a \leqslant x < x_0} u(x)$, and $u(\xi) \geqslant 0$. Since $u \in \mathfrak{W}$, we have $u'(\xi) = 0$ ana $u''(\xi) \leqslant 0$. Similary there exists a point $\eta \in (x_0, b]$ such that $u(\eta) \geqslant 0$, $u'(\eta) = 0$ and $u''(\eta) \leqslant 0$. Restricting ourselves to the interval (ξ, η), it follows that the function $v = L_2 u$ satisfies the system

$$\text{(2.5)} \qquad \begin{gathered} v''(x) + p_1(x) v'(x) + q_1(x) v(x) \geqslant 0, \quad x \in (\xi, \eta) \\ v(\xi) \leqslant 0, \quad v(\eta) \leqslant 0. \end{gathered}$$

From (2.5) and in view of the hypothesis concerning the function w, it follows from Theorem A that either $v = 0$ or $v < 0$ in (ξ, η). Since u has a nonpositive minimum value at $x_0 \in (\xi, \eta)$, we have $v(x_0) = (L_2 u)(x_0) \geqslant 0$, so $v = 0$ in (ξ, η).

Therefore we have

$$\text{(2.6)} \qquad \begin{gathered} r(x) u''(x) + p_2(x) u'(x) + q_2(x) u(x) = 0, \quad x \in (\xi, \eta) \\ u'(\xi) = 0, \quad u'(\eta) = 0. \end{gathered}$$

Since u attains a nonpositive minimum in $x_0 \in (\xi, \eta)$, then by the maximum principle for the second order equations, from (2.5) follows that $u = m$ in (ξ, η). We conclude that M is open relative to (a, b). Since $u(x) = m \leqslant 0$ for all $x \in [a, b]$ and $u(a) \geqslant 0$ and $u(b) \geqslant 0$, then we have $u = 0$ in $[a, b]$, and the proof is completed.

Theorem 2.1 can be generalized as follows:

THEOREM 2.2. *Let* $u \in \mathfrak{W}$ *satisfy the differential inequalities* (2.4), *where* L *is defined by* (2.1). *Suppose that the given functions* p_k, q_k $(k = 1, 2)$ *and* r *satisfy the following assumptions*: 1° $p_2, q_2, r \in C^2[a, b]$, $r(x) > 0$ *for* $x \in [a, b]$ *and there exists a function* $z \in \mathfrak{W}$ *satisfying*

$$\text{(2.7)} \qquad \begin{gathered} z(x) > 0, \quad x \in [a, b] \\ r(x) z''(x) + p_2(x) z'(x) + q_2(x) z(x) \leqslant 0, \qquad x \in (a, b) \\ z'(a) = 0, \quad z'(b) = 0 \end{gathered}$$

2° *the functions* p_1, q_1 *are bounded on every closed subinterval of* (a, b) *and there exists a function* $w \in C^2[a, b]$ *such that the inequalities* (2.3) *hold.*

Then u *cannot assume a nonpositive minimum value at an interior point of* (a, b) *unless* u *is identically zero.*

Proof. Let us denote by $y = u/z$, where u satisfies (2.4). It is easily seen that y satisfies the inequalities

$$\begin{aligned} &(L_1\tilde{L}_2 y)(x) \geqslant 0, \quad x \in (a, b) \\ &y(a) \geqslant 0,\ y(b) \geqslant 0 \text{ and } y'(a) \geqslant 0,\ y'(b) \leqslant 0 \end{aligned} \tag{2.8}$$

where

$$(\tilde{L}_2 y)(x) = \tilde{r}(x)y''(x) + \tilde{p}_2(x)y'(x) + \tilde{q}_2(x)y(x),$$

and

$$\begin{aligned} \tilde{r}(x) &= r(x)z(x) \\ \tilde{p}_2(x) &= p_2(x)z(x) + 2r(x)z'(x) \\ \tilde{q}_2(x) &= r(x)z''(x) + p_2(x)z'(x) + q_2(x)z(x). \end{aligned} \tag{2.9}$$

Suppose u assumes a nonpositive minimum value at a point $x_0 \in (a, b)$. We have $y(x_0) = u(x_0)/z(x_0)$ and $z(x_0) > 0$, then $y(x_0) \leqslant 0$. Since $y(a) \geqslant 0$ and $y(b) \geqslant 0$, so y assumes a nonpositive minimum value a point of (a, b).

On the other hand the function y satisfies the inequalities (2.8) and by (2.7) and (2.9) the operator $L_1\tilde{L}_2$ satisfies all the assumptions of Theorem 2.1. From this, by Theorem 2.1 we have $y = 0$ in $[a, b]$, and so $u(x) = y(x)z(x) = 0$ for all $x \in [a, b]$.

Remark 1. The results of Theorem 2.1 and Theorem 2.2 continue to hold if all the inequalities involving u are reversed, provided the words "nonpositive minimum" are replaced by the words "nonnegative maximum".

Remark 2. The following example shows that Theorem 2.1 as well Theorem 2.2 are false if the inequalities $u(a) \geqslant 0$ and $u(b) \geqslant 0$ are amitted.

Indeed, the function $u = \sin x - 16$ attains its negative minimum value at $x\frac{3}{2}\pi$ and yet satisfies the system

$$(Lu)(x) \equiv u^{(4)}(x) - \frac{8}{9}u''(x) - \frac{1}{9}u(x) \geqslant 0,\ x \in \left(\frac{\pi}{2}, \frac{5}{2}\pi\right)$$

$$u\left(\frac{\pi}{2}\right) = -15,\ u\left(\frac{5}{2}\pi\right) = -15,\ u'\left(\frac{\pi}{2}\right) = 0,\ u'\left(\frac{5}{2}\pi\right) = 0.$$

Let us observe that $L = L_1L_2$, where

$$L_1\varphi = \varphi'' + \frac{1}{9}\varphi,\ L_2\varphi = \varphi'' - \varphi,\ \varphi \in C^2\left(\frac{\pi}{2}, \frac{5}{2}\pi\right)$$

and the operators satisfy all assumptions of Theorem 2.1 in the interval $\left[\frac{\pi}{2}, \frac{5}{2}\pi\right]$. In fact, the function $w = \sin\left(\frac{1}{3}x\right)$ is positive in $\left[\frac{\pi}{2}, \frac{5}{2}\pi\right]$ and $L_1 w = 0$ in $\left[\frac{\pi}{2}, \frac{5}{2}\pi\right]$.

Remark 3. If in the inequalities (2.4) we have $u(a) = u(b) = 0$ instand of the inequalities $u(a) \geqslant 0$ and $u(b) \geqslant 0$, then the assumption $z'(a) = z'(b) = 0$ in Theorem 2.2 (see (2.7)) can be omitted.

As a consequence of Theorem 2.2, we obtain an a priori estimate from which readily follows the uniqueness and the continuous dependence solutions.

COROLLARY 2.1. *Suppose that the boundary value problem*

$$(2.10)\qquad \begin{gathered}(Lu)(x) = f(x), \quad x \in (a, b)\\ u(a) = \gamma_1,\ u'(a) = \gamma_2,\ u(b) = \gamma_3,\ u'(b) = \gamma_4\end{gathered}$$

can be solved for arbitrary continuous function f and arbitrary constants $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. If the operator L is defined by (2.1) *and satisfies the assumptions of Theorem* 2.2, *and if u is a solution of* (2.10), *then for all $x \in [a, b]$*

$$(2.11)\qquad |u(x)| \leqslant c \max\{\max|f|, \max(|\gamma_1|, \gamma_2|, |\gamma_3|, |\gamma_4|)\},$$

where the positive constant c depends only on the coefficients p_k, q_k $(k = 1, 2)$ and r.

The proof of Corollary 2.1 is the same as the proof of Corollary 2 in paper [2] and is omitted.

3. Application of Theorems 2.1 and 2.2 to the inequalities with constant coefficients

Let us consider a general ordinary differential operator with real constant coefficients of the fourth order, i.e.,

$$(3.1)\qquad L\varphi = \varphi^{(4)} + c_1\varphi''' + c_2\varphi'' + c_3\varphi' + c_4\varphi,\ \varphi \in \mathfrak{W}$$

and $c_1, c_2, c_3, c_4 \in R$.

It is easily verified that the operator L defined by (3.1) can be expressed in the form $L = L_1L_2$, where

$$(3.2)\qquad L_k\varphi = \varphi'' + \alpha_k\varphi' + \beta_k\varphi,\ \varphi \in C^2(a, b), \quad k = 1, 2$$

and $\alpha_k, \beta_k \in R$, $k = 1, 2$.

Since the operators L_1 and L_2 commute and $\beta_1\beta_2 = c_4$, as a consequence of Theorem 2.1 we obtain the following

THEOREM 3.1. *If the coefficient c_4 of the operator* (3.1) *is nonpositive, then to this operator can be applied the maximum principle formulated in Theorem* 2.1.

Proof. From the assumption $c_4 \leqslant 0$ and from the equality $\beta_1\beta_2 = c_4$ follows that $\beta_1 \leqslant 0$ and $\beta_2 \geqslant 0$ or $\beta_2 \leqslant 0$ and $\beta_1 \geqslant 0$. Since L_1 and L_2 commute, we can take that $\beta_2 \leqslant 0$ and $\beta_1 \geqslant 0$. It is easily seen there exists a interval $[a, b]$ for which exists a function $w \in C^2[a, b]$ such that

$$\begin{gathered}w(x) > 0, \quad x \in [a, b]\\ w''(x) + \alpha_1 w'(x) + \beta_1 w(x) \leqslant 0, \quad x \in (a, b)\end{gathered}$$

and so the operator L satisfies all assumptions of Theorem 2.1.

THEOREM 3.2. *If the coefficient* c_4 *of the operator* (3.1) *is positive, then if it comes to the worst, we can apply the maximum principle formulated in Theorem* 2.2 (*see also Remark* 3) *to the operator* L.

The proof of Theorem 3.2, as a simple consequence of above considerations, is omitted.

Remark 4. The interval $[a, b]$ in which can be applied the maximum principle for the operator L of the type (3.1) always exists if only number $(b-a)$ is not large depending on the coefficients c_1, c_2, c_3, c_4 of the operator L.

References

[1] S. N. Chow, D. R. Dunninger and A. Lasota, *Maximum principle for fourth order ordinary differential equations*, J. Diff. Equations, 14 (1973), 101—105.

[2] D. R. Dunninger, *Maximum principle for fourth order ordinary differential inequalities*, J. Math. Anal. Appl., 82 (1981), 399—405.

[3] M. H. Proffer and H. F. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, N. J. 1967.

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