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On the existence of a subanalytic selection for the subanalytic set-valued function

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The aim of this paper is to prove that for a set-valued function which is subanalytic, a subanalytic selection can be chosen. This fact has turned out to be useful as a tool in optimal control, where subanalytic and semianalytic set-valued functions appear, for instance, in the works of H. J. Sussmann, R. J. Aumann, H. Halkin, E. C. Hendricks etc.

THEOREM. *Let W, V be two finite-dimensional vector spaces, $E \subset W \times V$ be a subanalytic set (not necessarily relatively compact) and F — the image of E by the natural projection π_W: $W \times V \to W$. Then there exists a subanalytic function φ: $F \to V$ such that $\varphi \subset E$.*

Remark. Let us notice that such a set E defines, in a natural way, a set-valued function

$$F \ni w \to E_w = \{v \in V \colon (w, v) \in E\}.$$

In the case where E is semianalytic and compact, this function is what Halkin and Hendricks ([5]) call a set-valued function of class H.

Proof of the theorem. Let us first assume that the set E is relatively compact. Suppose $W = R^n$, $V = R^m$. We will proceed by induction on m. Assume $m = 1$. By Lemma 1 of the paper [4] and the theorem of the paper [3], the functions

$$\Phi\colon F \times V \ni (w, v) \to \varrho(v, E_w) \in R,$$

$$\Psi\colon F \times V \ni (w, v) \to \varrho(v, (G \setminus E)_w) \in R$$

where ϱ is the Euclidean metric and $G = F \times (-c, c)$ (c being a positive constant such that $E \subset \{-c < v < c\}$), are subanalytic. Let us take the set $X = \Phi^{-1}(\{0\}) \cap \Psi^{-1}(\{0\})$. In view of the subanalicity of Φ and Ψ, this set is subanalytic in $W \times V$ ([2]). Let us take the function

$$h\colon (W \times V)^2 \ni ((w, v), (\overline{w}, \overline{v})) \to \left(w, \frac{v - \overline{v}}{2}\right) \in W \times V$$

and then the set $Z = h(X \times X \cap \{w = \overline{w}\}) \cap E$. Observe that Z is subanalytic ([2]) and relatively compact. The function φ defined as follows:

$$\varphi\colon F \ni w \to \min Z_w \in V$$

will be the desired selection. It is easy to see that φ is a well-defined function (Z_w being finite and nonempty ([2])), $\varphi \subset E$ and φ is subanalytic ([2]) because $\varphi = \{\Phi(w, v) - v - c = 0\}$. Now let us assume the theorem true for m such that $m < k$ and let $m = k$. We already know that there exists a subanalytic function $\psi\colon \pi_{W \times R}(E) \to R^{k-1}$ such that $\psi \subset E$ and a subanalytic function $\eta\colon F \to R$ such that $\eta \subset \pi_{W \times R}(E)$ (where $\pi_{W \times R}\colon W \times V \ni (w_1, \ldots, w_n, v_1, \ldots, v_k) \to (w_1, \ldots, w_n, v_1) \in W \times R$). We take $\varphi = \psi \circ (\mathrm{id}_F, \eta)$ and this will be the desired selection because $\varphi \subset E$ and φ is subanalytic as a composition of subanalytic functions of the relatively compact graphs (see [2]).

Now, if E is not relatively compact one can take concentric cubes $K_1 \subset K_2 \subset \ldots$ such that $\bigcup_{i=1}^{\infty} K_i = W \times V$. We than take the subanalytic and relatively compact set $E \cap K_1$ and the existing subanalytic selection $\varphi_1\colon \pi_w(E \cap K_1) \to V$, $\varphi_1 \subset E \cap K_1$. Next, for each $i = 2, 3, \ldots$ we can take the restriction of the existing subanalytic selection, namely $\varphi_i\colon \pi_W(E \cap K_i) \setminus \pi_W(E \cap K_{i-1}) \to V$. Thus φ_i is subanalytic (as a restriction to the subanalytic set), $\varphi_i \subset E \cap K_i$. Let us observe that $\bigcup_{i=1}^{\infty} \varphi_i$ is a function contained in E and subanalytic as a union of a locally finite family of subanalytic sets.

References

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