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Remarks on G. S. Jones paper
The existence of critical points in generalized dynamical systems

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The purpose of this notice is to discuss the statements and proofs presented in the paper [1] by Jones. The exact text from [1] is *italicized* (only t_1, t_2 are replaced by t, s here).

The idea of this paper arose when studying [1] for presentation in the seminar of Professor A. Pelczar in 1981).

1. Definitions, assumptions and conventions

Denote $[0, \infty)$ by R^+ and $[0, 1]$ by I.

Let X be a closed locally compact convex subset of a Banach space B, and let π: $R^+ \times X \to X$. Let S be a set in X. If for some $t > 0$, $\pi(t, \bar{S}) \subset S$, then S is said to be ***constrained*** by π (this notion is never used either in [1] or here). *If for some $t > 0$, $\pi(t, \bar{S})$ is contained in a compact subset of S* (we denote it by U_t), *then S is said to be* ***compactly constrained*** (CC) *by π at t. If for each $x \in \bar{S}$ there exists some $t > 0$ such that $\pi(t, x) \in S$, then S is said to be weakly constrained by π. $\pi(\infty, x)$ will denote the set of limit points of all sequences $\{\pi(t_n, x)\}$ where $\{t_n\} \subset R^+$ is any sequence tending to ∞. For $S \subset X$, $\pi(\infty, S) = \bigcup\{\pi(\infty, x): x \in S\}$.* If $x \in X$ is such that for each $t \in R^+$, $\pi(t, x) = x$, then x is said to be a critical point of π.

It is also convenient to define ϕ to be a function $\phi: I \times R^+ \to R^+$ which is positive except possibly on $I \times \{0\}$ and strictly increasing on its second argument.

In [1] the term "π compactly constrains S" is also used. One can guess it means that π CC S at some t. But as we will soon see a stronger condition is to be used. In this paper we will say that π CC′ S if for some t_0, $\pi(t, \bar{S})$ is contained in a common compact set $U \subset S$ for all $t > t_0$.

π is said to be a generalized dynamical system (GDS), iff $\pi: R^+ \times X \to X$ *is a continuous mapping such that* $\pi(0, x) = x$, *and* $\pi(t, \pi(s, x)) = \pi(t+s, x)$ *for all* $x \in X$, t, $s \in R^+$ [1].

2. Tychonov Fixed Point Theorem ([3], or [2], p. 15).

If $U \neq \emptyset$ is a compact convex set in a locally convex space, $T: U \to U$ is a continuous map, then T has a fixed point.

3. Lemmas

LEMMA 1. *Let S be an open set in X and let $\pi: R^+ \times X \to X$ be a generalized dynamical system which compactly constrains S at t_0. Then there exists $t_1 > 0$ such that $\pi(t, \bar{S}) \subset S$ for all $t \geqslant t_1$.*

The proof in [1] seems to be correct, but the restriction to GDS is essential in it. It is used now in studying mappings q (in Lemmas 3 and 4), which are not GDS. So our main objective will be to make these Lemmas independent of Lemma 1. This is the purpose of introducing CC' here.

LEMMA 2. *Let S be an open convex subset of X, let $f: I \times X \to X$ be a continuous function, and let $f(0, \cdot)$ have a fixed point in S. Then either $f(1, \cdot)$ has a fixed point in S or for every compact subset U of S there exists $\lambda \in I$ such that $f(\lambda, \cdot)$ has a fixed point in $S \setminus U$.*

The proof has been found incomplete. No help was found in the paper of Dugundji [4] indicated in [1], either. The attempts to prove the Lemma led to the following counterexample:

$$f(\lambda, x) = x - x^2 - \lambda \quad (S = X = R)$$ [2]

(Use $U = [-2, 2]$ to disprove the thesis.)

A slightly modified version of this Lemma can be, however, proved and it will do in the proof of the Lemma 3 (but not in the Theorems 2 and 3):

LEMMA 2'. Let S be an open convex set in X and let $f: I \times X \to X$ be a continuous function. Let U be a compact subset of S such that

(a) $f(0, \bar{S}) \subset U$

(b) for all $t \in I$, $f(t, \cdot)$ has no fixed point in $\bar{S} \setminus U$.

Then $f(1, \cdot)$ has a fixed point in S. The proof will be based on the

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[1] The definition of GDS was given so late here — whereas in [1] it comes just after the introduction, of π — because compactly and weakly constraining are used afterwards for mappings which are not GDS. However in [1] these notions are defined only for π being GDS.

[2] The way of using this lemma in Theorems 2 and 3 requires replacing "$\lambda \in I$" in the thesis by "$\lambda \in I \setminus \{0\}$", and then $f(\lambda, x) = x - \lambda$ is a good counterexample, even with an extra assumption that $f(0, x) = x$ for all $x \in X$.

Browder-Potter's Theorem (in [2], p. 25—31)

If M is a closed convex set in a normed space B, K is a compact set in B, $u\colon M\times I\to K$ is continuous, and for $u_t = u(\cdot, t)\colon x\to u(x,t)$ we have:

(A) $u_0(\partial M)\subset M$ where ∂M is the boundary of M,

(B) for each $t\in I$ u_t has no fixed point on ∂M,

then u_1 has a fixed point in M.

One can see that u_1 must have its fixed point in int M as from (B) u_1 has no fixed point on ∂M.

To prove Lemma 2' let us take a compact convex set $M\subset\bar{S}$ such that $U\subset \operatorname{int} M$ [3]. For $t\in I$, $x\in M$ we define $u(x,t) = f(t,x)$. $u\colon M\times I\to K$ (where $K = f(I\times M)$ is compact) satisfies the assumptions of Browder-Potter's theorem:

$$u_0(\partial M) = f(0,\partial M)\subset f(0,\bar{S})\subset U\subset M$$

and for all $t\in I$, u_t, has no fixed point on ∂M (from (b): $\partial M\subset \bar{S}\setminus U$). So $u_1 = f(1,\cdot)$ has a fixed point in $M\subset\bar{S}$ and even in $\operatorname{int} M\subset S$.

LEMMA 3. *Let S be an open convex set contained in X, let $\psi\colon R^+\to R^+$ be a positive strictly increasing function on $R^+\setminus\{0\}$* [4]*, and let $q\colon R^+\times X\to X$ be a continuous map such that $q(0,x) = x$ and for all t and s in R^+, $q(t, q(s,\bar{S}))\subset q(t+\psi(s),\bar{S})$* [5]*. If q constrains S compactly for all t sufficiently large* [6]*, then $q(t,\cdot)$ has a fixed point for every $t\in R^+$.*

Proof. *There exists t^* such that for all $t\geqslant t^*$, $q(t,\bar{S})$ is contained in a compact subset U of S* (note that U is assumed to be common for all $t\geqslant t^*$. This will be of importance later on). *From the Tychonov fixed point theorem* the thesis holds for $t\geqslant t^*$.

The second part of the proof is also based on [1], with some redaction changes.

Let us suppose that for some $t_0\in(0,t^*)$, $q(t_0,\cdot)$ has no fixed point in S. Then by Lemma 2' with U as above and $f(t,x) = q(t^*-t(t^*-t_0),x)$ (so $f(0,\cdot) = q(t^*,\cdot)$, $f(1,\cdot) = q(t_0,\cdot)$) — there exists $t'\in I$ such that $f(t',\cdot)$ has a fixed point in $\bar{S}\setminus U$, i.e. there exists $t\in[t_0,t^*]$ such that $q(t,\cdot)$ has a fixed point $x_1\in\bar{S}\setminus U$. Similarly as in [1], we will show that this is impossible. Of course from $x_1 = q(t,x_1)$ we get that $x_1 = q^n(t,x_1)$ where q^n denotes the n-th iteration of $q(t,\cdot)$. From the assumptions we have that $q(t,q(t,\bar{S}))\subset q(\psi(t)+t,\bar{S})$ and applying $q(t,\cdot)$ to it we obtain

$$q^3(t,\bar{S})\subset q(t,q(\psi(t)+t,\bar{S}))\subset q(\psi(t)+\psi(t)+t,\bar{S}).$$

A simple induction gives that $q^n(t,\bar{S})\subset q(t+(n-1)\cdot\psi(t),\bar{S})$ and it is contained in U for n sufficiently large: such that $(n-1)\cdot\psi(t)\geqslant t^*$ — here one can see the necessity of U being common at least for all $\tau\in[t^*, t^*+\psi(t^*)]$; in [1] no one word was said about it.

[3] The existence of such M can be proved by using the premises that X is locally compact, S is open, U is compact, S (so also $\bar{S}$) is convex, and for any compact set W in a Banach space the set $\overline{\operatorname{conv} W}$ is compact too (see [5] Theorem 3.25 (a) on p. 72—73).

[4] The word "positive" should be omitted — it is rather misguiding.

[5] Or rather $q(t,q(s,\bar{S}))\subset q(\psi(t)+s,\bar{S})$.

[6] Or rather q CC' S.

Also, it is clear now why the footnote 5 was given. Now, $x_1 = q^n(t, x_1) \in U$, but $x_1 \in \bar{S} \setminus U$ — and we have a contradiction.

LEMMA 4. *Let S be an open set contained in X, ψ: $R^+ \to R^+$ be a positive strictly increasing function on $R^+ \setminus \{0\}$* [7], *and let q: $R^+ \times X \to X$ be a continuous map such that $q(0, x) = x$ and for all t and s in R^+, $q(t, q(s, \bar{S})) \subset q(t+\psi(s), \bar{S})$* [8]. *Furthermore, let J be a closed interval $[0, b]$, $b > 0$* [9], *and let $q(t, \cdot)$ have a fixed point for each $t \in J$. If for all t and s in J such that $t+s \in J$ we have that $q(t+s, x) = q(t, q(s, x))$ for all x in $\bar{S}$, then there exists x^* in S such that $q(t, x^*) = x^*$ for all $t \in J$.*

The proof presented in [1] is almost clear, but it uses the assumption that q CC S, so this assumption should be added in the Lemma (or better q CC' S).

4. Theorems

THEOREM 1. *Let S be an open convex set in X and let π: $R^+ \times X \to X$ be a GDS. If π* CC *S* (or rather π CC' S) *then π has a critical point*

Proof. We have now Lemmas 3 and 4; to make them work here, CC is changed into CC'. Then, the proof from [1] can be followed.

From Lemma 3, with $q = \pi$ and $\psi(t) = t$, *it follows that $\pi(t, \cdot)$ has a fixed point for each $t \in R^+$. Hence by Lemma* 4 *with $J = R^+$ it follows that there exists x^* such that $\pi(t, x^*) = x^*$ for all $t \in R^+$.*

THEOREM 2. *Let S be an open convex set in X and let π: $R^+ \times X \to X$ be a GDS. Let $\pi(\infty, \bar{S})$ be a compact subset of S, let r be a retraction of X onto the closed convex hull H of $\pi(\infty, \bar{S})$* [10], *and for all $\lambda \in I$ let us define*

$$\pi_\lambda(t, x) = (1-\lambda)\cdot\pi(t, x) + \lambda\cdot r\pi(t, x).$$

If for all t and s sufficiently large,

$$\pi_\lambda(t, \pi_\lambda(s, \bar{S})) \subset \pi_\lambda(t+\phi(\lambda, s), \bar{S})$$

(where ϕ is some function as on *p.* 1),
then π has a critical point in S.

A counterexample will be given below. The proof in [1] contains an error on p. 16, verse 2.

Next theorem is false too.

THEOREM 3. *Let S be an open set in X and let π: $R^+ \times X \to X$ be a GDS. Suppose there exists a continuous function ζ: $I \times R^+ \times X \to X$ such that for all $\lambda \in I$ and all t, s in R^+,*

$$\zeta(\lambda, t, \zeta(\lambda, s, \bar{S})) \subset \zeta(\lambda, t+\phi(\lambda, s), \bar{S})$$

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[7] The word "positive" should be omitted again.

[8] Or rather $q(t, q(s, \bar{S})) \subset q(\psi(t)+s, \bar{S})$ — again.

[9] The interval $[0, \infty)$ should be also admitted here for some convenience in the proof of the theorem 1.

[10] I.e. r: $X \to H$ is continuous, and for $h \in H$, $r(h) = h$.

(where ϕ is some function as on p. 1), and $\zeta(0,\cdot,\cdot) = \pi$. *If for each $\lambda \in I$, $\zeta(\lambda,\cdot,\cdot)$ weakly constrains S, $\zeta(1,\cdot,\cdot)$ CC S for all t_0 sufficiently large, and $\bigcap \{\zeta(\lambda, \infty, \bar{S}) \colon \lambda \in I\}$* [11] *is compact, then π has a critical point.*

The proof in [1] contains an error on p. 17, verse 5. Moreover, it is based on Lemma 2, and on Lemma 4 with $q = \pi$, which is proved only with an extra assumption that q CC' S. Such an assumption would make Theorem 3 a special case of Theorem 1.

5. Counterexample

Let us first construct a GDS π_0 in a semiplane $\{(\varrho, z) \colon \varrho \geqslant 0\}$ as in the picture

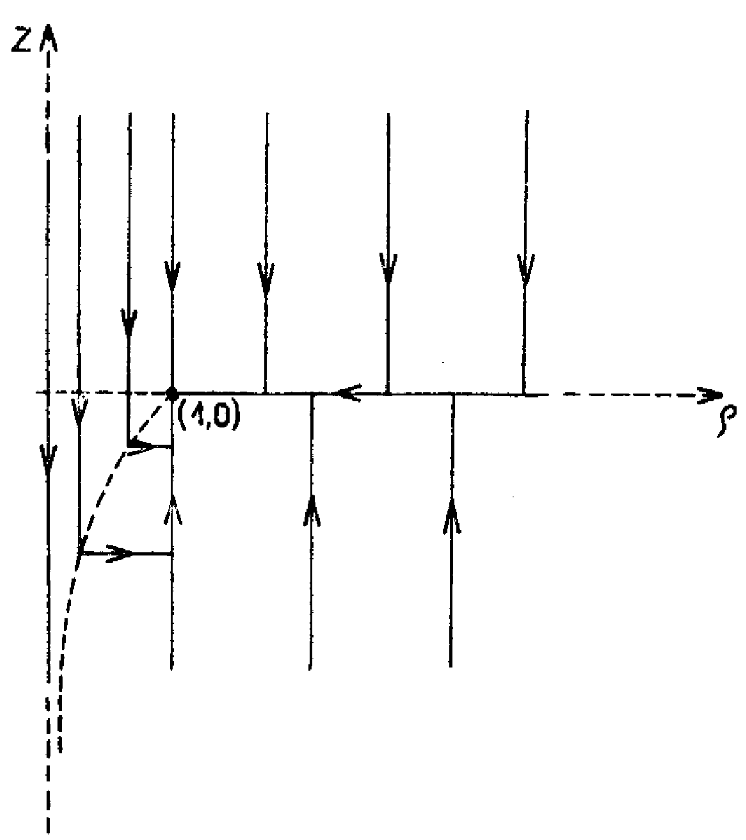

Fig. 1

Everywhere the absolute value of the velocity is 1, except the critical point $(1,0)$ (and points where the trajectories are not smooth). One can easily see that π_0 defined in this way is continuous.

π_0 can be formally defined by steps like this:

$$\pi_2(t,(\varrho,z)) = \begin{cases} (\varrho, z+t) & \text{when } \varrho \geqslant 1,\ t+z \leqslant 0 \\ \pi_1(t+z,(\varrho,0)) & \text{when } \varrho \geqslant 1,\ z \leqslant 0 < t+z \end{cases}$$

where π_1 is previously defined for $z = 0$, $\varrho \geqslant 1$.

Now we will define a GDS π in $X = R^3$ (and $S = R^3$ in the sequel):

$$\pi(t,(\varrho,z,\varphi)) = (\pi_0(t,(\varrho,z)), \varphi+t),$$

[11] The version from p. 9 in [1] of this theorem is taken here. On p. 16 there is $\bigcap$ instead of $\bigcup$, which is less sensible.

where (ϱ, z, φ) is a representation of the point $(x, y, z) \in X$ with $x = \varrho\cos\varphi$, $y = \varrho\sin\varphi$, $z = z$.

π can be treated as a product of π_0 and a rotation dynamical system.

This GDS can be used to disprove Theorems 2 and 3. $S = X$ is an open convex set. $\pi(\infty, \bar{S})$ is a circle (for points on the axis z the set $\pi(\infty, (0, 0, z))$ is empty), H is the disc defined by $z = 0$, $\varrho \leqslant 1$.

The retraction r will be defined by

$$r(\varrho, z, \varphi) = \begin{cases} (\varrho, 0, \varphi) & \text{when } \varrho < 1 \\ (1, 0, \varphi) & \text{when } \varrho \geqslant 1 . \end{cases}$$

With π_λ defined as in Theorem 2 we should check the inclusion $\pi_\lambda(t, \pi_\lambda(s, \bar{S})) \subset \pi_\lambda(t+\phi(\lambda, s), \bar{S})$. We put $\phi(\lambda, s) = s$ (however any function ϕ with properties as on p. 1 can be taken). Let us distinguish two cases:

a) $\lambda \neq 1$. We will show that $\pi_\lambda(t, \bar{S}) = X$ for every t, so the inclusion is trivial: its right side is the whole space. Fix $t \in R^+$ and $y \in X$. To show that $y \in \pi_\lambda(t, \bar{S})$ we take $y_0 = r(y) + \frac{1}{1-\lambda}(y - r(y))$, and then we take a point x such that $\pi(t, x) = y_0$ (it exists, of course). Now $\pi_\lambda(t, x) = y$.

b) $\lambda = 1$, so $\pi_\lambda = \pi_1 = r \circ \pi$. Now, of course, the left side of the inclusion is contained in H. We will show that $\pi_1(t, \bar{S}) \supset H$ for every t. For fixed $y \in H$ and $t \in R^+$ we take a point x such that $\pi(t, x) = y$ (x need not be from H). Then $\pi_1(t, x) = y$ too.

As we can see now, all the conditions of Theorem 2 are satisfied, but π has no critical point.

Define, in Theorem 3, $\zeta(\lambda, t, x) = \pi_\lambda(t, x)$. We obtain that $\zeta(0, \cdot, \cdot) = \pi$, $\zeta(\lambda, \cdot, \cdot)$ weakly constrains S (because $S = X$), $\zeta(1, \cdot, \cdot)$ CC' S (so also CC S for large t_0), because $\zeta(1, R^+, \bar{S}) \subset H$ — a compact subset of S. We will show now that

$$\bigcup\{\zeta(\lambda, \infty, \bar{S}) \colon \lambda \in I\} = \{(\varrho, z, \varphi) \colon z = 0,\ \varrho = 1 \text{ or } 0\}$$

(a circle with its centre).

Fix $\lambda \in I$ and $x \in R^3$. If x is not on axis z, then after a finite time $\pi(t, x)$ reaches the circle and starts to move along it; then $\zeta(\lambda, t, x) = \pi_\lambda(t, x) = \pi(t, x)$, and $\zeta(\lambda, \infty, x)$ is the circle.

If x is taken from the z axis, then $r \circ \pi(t, x) = (0, 0, 0)$ for all t, so for $\lambda \neq 1$, $\zeta(\lambda, t, x) = \lambda\pi(t, x)$ tends to infinity ($\zeta(\lambda, \infty, x)$ is empty) and for $\lambda = 1$, $\zeta(\lambda, t, x) = r \circ \pi(t, x) = (0, 0, 0)$ for all t, and then $\zeta(\lambda, \infty, x) = \{(0, 0, 0)\}$. In this way all the conditions of the Theorem 3 are satisfied, but π has no critical point.

6. A more regular counterexample

An example of C^1 dynamical system (i.e. $\pi\colon R \times X \to X$ with the conditions of GDS valid in the whole R) can be obtained by replacing π_0 in the part 5 of this paper by the dynamical system shown in the picture:

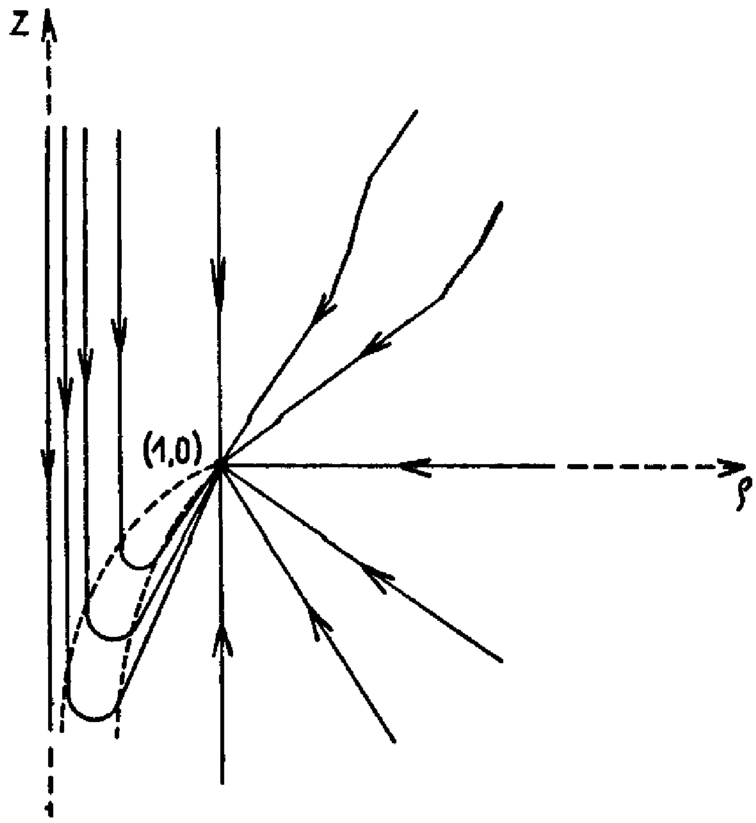

Fig. 2

The absolute value of the velocity in a point (ϱ, z) is equal to the distance between (ϱ, z) and $(1, 0)$. The GDS π in $X = R^3$ is obtained from π_0 as in the part 5 — by rotating around the axis z.

References

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