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Dependence on parameter of solution of a non-linear problem

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Abstract. Let X be a Hilbert space, let Y be a Banach space and let $\mathcal{U}$ be an open subset of a Banach space. Let $A_h\colon X \to Y$ be a linear, closed, dense defined operator with closed range and closed kernel, for $h \in \mathcal{U}$. The main object of this paper is to study the differentiability with respect to the parameter h of solutions of the equation $A_h u = G_h u$, where $G_h\colon X \to Y$ is a non-linear operator, for $h \in \mathcal{U}$.

§ **1.** In this part X is a real Banach space and $\mathcal{U}$ is an open subset of a metric space. For any $h \in \mathcal{U}$, X_h is a closed subset of X. A family of mappings

$$K_h\colon X_h \to X_h$$

is called uniformly contractive in $\mathcal{U}$ if there exists $\delta \in (0, 1)$ such that

$$||K_h(x) - K_h(y)|| \leqslant \delta ||x-y|| \quad \text{for } h \in \mathcal{U},\ x, y \in X_h.$$

According to the Banach contraction principle, every K_h, $h \in \mathcal{U}$, has a unique fixed point $x_h \in X_h$.

LEMMA 1. *If the mapping*

$$\mathcal{U} \ni h \to K_h(0) \in X$$

is bounded and K_h is uniformly contractive, then

a) *the sequence $K_h(0)$, $K_h^2(0) = K_h(K_h(0))$, ... of iterates converges uniformly in $\mathcal{U}$ and*

b) *the mapping*

$$\mathcal{U} \ni h \to x_h \in X$$

is bounded.

Proof. Let K_h^ν, $\nu = 1, 2, \ldots$, be the sequence of iterates of K_h. Uniform convergence of the sequence $\{K_h^\nu(0)\}$ follows from the estimation (see e.g. [2], p. 10)

$$||K_h^\nu(0) - K_h^{\nu+p}(0)|| \leqslant \frac{\delta^\nu}{1-\delta} ||K_h(0)||$$

for every positive integers ν, p.

Now letting $\nu = 1$ and $p = \mu - 1$ we obtain

$$\|K_h^{\mu}(0)\| \leqslant \frac{1}{1-\delta}\|K_h(0)\|.$$

Since $K_h^{\mu}(0) \to x_h$ as $\mu \to \infty$, we have

$$\|x_h\| \leqslant \frac{1}{1-\delta}\|K_h(0)\|.$$

§ **2.** Let X be a Hilbert space and assume that Y is a Banach space and D is a dense subspace of X. We shall consider a family of linear operators

$$A_h : D \to Y, \qquad h \in \mathcal{U},$$

where $\mathcal{U}$ is an open subset of a Banach space $\mathcal{B}$. We shall assume that

1° A_h is a closed operator and

2° the range $R(A_h)$ and the kernel $N_h = \operatorname{Ker} A_h$ of the operator A_h are closed subsets of Y and X, respectively, for $h \in \mathcal{U}$.

Let $X_h = N_h^{\perp}$ be the orthogonal complement of N_h. Then X_h as a closed subset of the Banach space X is complete. Thus it is a Banach subspace of X and the restriction

$$A_h|D \cap X_h \colon D \cap X_h \to R(A_h)$$

is a closed invertible linear operator. Therefore, by the closed graph principle, its inverse

$$(A_h|D \cap X_h)^{-1} \colon R(A_h) \to X_h$$

is a continuous operator. Let $T_h \colon Y \to X_h$ be an extension of $(A_h|D \cap X_h)^{-1}$ with the same norm. Let us consider, for $h \in \mathcal{U}$, the orthogonal projection

$$P_h \colon X = N_h \oplus X_h \ni (x' + x'') \to x' \in N_h,$$

where $x' \in N_h$, $x'' \in X_h$ and the sum "$\oplus$" is direct. It is clear that for $x \in D$ we have

$$T_h A_h x = x - P_h x.$$

Thus $T_h A_h$ is a bounded dense defined linear operator, for all $h \in \mathcal{U}$.

We shall consider a family of operators

$$\mathcal{U} \times \mathcal{U} \ni (w, h) \to T_w A_h \colon X \to X_w \subset X$$

and we shall assume that $T_w A_h \in B(X)$ for all $(w, h) \in \mathcal{U} \times \mathcal{U}$, where $B(X)$ is the Banach algebra of linear bounded operators from X to X. Let $[T_w A_h]$ be defined as the unique extension of $T_w A_h$ to all of X in such a way that the extension has the same norm.
Let

$$G \colon \mathcal{U} \times D \ni (h, x) \to G_h(x) \in Y$$

be a family of operators with domain D for all $h \in \mathcal{U}$.
We shall study differentiability with respect to the parameter h of the solution x_h of the equation

$$A_h x = G_h x, \qquad h \in \mathcal{U}. \tag{1}$$

For this purpose the following assumptions will be indispensable in the sequel: Let a be a fixed point of $\mathscr{U}$.

Assumption Z_1. *There exist* $\delta \in (0, 1)$ *and an open neighborhood* V *of* a *such that*

$$||T_h G_w x - T_h G_w y|| \leqslant \delta ||x-y||$$

for all $(h, w) \in ((\{a\} \times V) \cup \{(h, w) \in V \times V\colon h = w\}$, $x, y \in D$.

Assumption Z_2. *The mapping*

$$\mathscr{U} \ni h \to T_h G_h(0) \in X$$

is bounded in an open neighborhood of a.

Assumption Z_3. *There exists an open neighborhood* $\mathscr{U}_a$ *of the point* a *such that for every* $h \in \mathscr{U}_a$ *the operator* $[T_a A_h] \in B(X)$ *and the mapping*

$$\mathscr{U}_a \ni h \to [T_a A_h] \in B(X)$$

is differentiable at a, *i.e. there exists a bounded linear operator* $W_a\colon B \to B(X)$ *such that*

$$|h|^{-1} ||[T_a A_{a+h} - T_a A_a] - W_a h|| \to 0, \quad \text{as } h \to 0,$$

where $|\cdot|$ *and* $||.||$ *are norms in suitable spaces.*

Assumption Z_4. *The mapping*

$$\mathscr{U} \ni h \to P_h \in B(X)$$

is differentiable at a. *The differential of this mapping at* a *will be denoted by* B_a.

Assumption Z_5. *Assumption* Z_1 *fulfils and if* $x_a \in X_a$ *is a solution of the equation* $[T_a A_a]x = [T_a G_a]x$, *then the mapping*

$$V \times X \ni (h, x) \to [T_a G_h](x) \in X$$

is differentiable at (a, x_a), *where* $[T_a G_h]$ *is defined as the unique extension of* $T_a G_h$ *to all of* X *in such a way that the extension satisfies the same Lipshitz condition (see e.g.* [6], p. 28)

Let C and D be the partial differentials of $[T_a G]$ with respect to h and x, respectively Then

$$d_{(a, x_a)} T_a G(h-a, x-x_a) = C(h-a) + D(x-x_a) \quad \text{for } h \in V,\ x \in X$$

Remark 1. Suppose that Assumption Z_1 is fulfilled. Then, by the Banach contraction theorem, there exists exactly one $x_h \in X$ such that $x_h = [T_h G_h] x_h$. Since $R(T_h) \subset X_h$, the point $x_h \in X_h$.

Remark 2. If assumptions Z_1, Z_2 hold, then it follows from Lemma 1 that the mapping

$$\mathscr{U} \ni h \to x_h \in X$$

is bounded in an open neighborhood of a.

Remark 3 It follows from Assumption Z_5 that
1° $\|D\| \leqslant \delta < 1$. Therefore we have
2° $I-D$ is an invertible element of $B(X)$.

THEOREM 1. *Under the assumptions Z_1—Z_5, let x_h be the solution of the equation $[T_a A_h]x = [T_a G_h]x$ for $h \in \mathscr{U}$. Then the mapping*

$$\mathscr{U} \ni h \to x_h \in X$$

is differentiable at a.

Proof. First we prove that the mapping $h \to u_h$ is continuous at the point a. For Δh small enough we have

$$\begin{aligned}
& x_{a+\Delta h}-x_a = (T_a A_a - P_a)(x_{a+\Delta h}-x_a) = T_a A_a x_{a+\Delta h} - T_a A_a x_a - \\
& -P_a x_{a+\Delta h} + P_a x_a = T_a A_a x_{a+\Delta h} - T_a G_a x_a - P_a x_{a+\Delta h} + \\
& +P_{a+\Delta h} x_{a+\Delta h} = T_a A_a x_{a+\Delta h} - T_a A_{a+\Delta h} x_{a+\Delta h} + \\
& +T_a A_{a+\Delta h} x_{a+\Delta h} - T_a G_a x_a + (P_{a+\Delta h}-P_a) x_{a+\Delta h} = \\
& = (T_a A_a - T_a A_{a+\Delta h}) x_{a+\Delta h} + T_a G_{a+\Delta h} x_{a+\Delta h} - T_a G_a x_a + \\
& +(P_{a+\Delta h}-P_a) x_{a+\Delta h} = (T_a A_a - T_a A_{a+\Delta h}) x_{a+\Delta h} + T_a G_{a+\Delta h} x_{a+h} - \\
& -T_a G_{a+\Delta h} x_a + (T_a G_{a+\Delta h} - T_a G_a) x_a + (P_{a+\Delta h}-P_a) x_{a+\Delta h}\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|x_{a+\Delta h}-x_a\| \leqslant \|T_a A_a - T_a A_{a+\Delta h}\|\,\|x_{a+\Delta h}\| + \delta\|x_{a+\Delta h}-x_a\| + \\
+\|(T_a G_{a+\Delta h} - T_a G_a) x_a\| + \|(P_{a+\Delta h}-P_a)\|\,\|x_{a+\Delta h}\|\,.
\end{aligned}$$

Thus, by the assumptions and Remark 2. the mapping $\mathscr{U} \ni h \to x_h \in X$ is continuous at a. To prove differentiability let us compute

$$\begin{aligned}
& |\Delta h|^{-1}\|x_{a+\Delta h}-x_a+(W_a\Delta h)x_a - C\Delta h - D(x_{a+\Delta h}-x_a) - (B_a\Delta h)x_a\| = \\
& = |\Delta h|^{-1}\|(T_a A_a - T_a A_{a+\Delta h})x_{a+\Delta h} + T_a A_{a+\Delta h} x_{a+\Delta h} - T_a G_a x_a + \\
& +(P_{a+\Delta h}-P_a)x_{a+\Delta h} + (-W_a\Delta h)x_a - C\Delta h - D(x_{a+\Delta h}-x_a) - (B_a\Delta h)x_a\| \leqslant \\
& \leqslant |\Delta h|^{-1}\|T_a A_a - T_a A_{a+\Delta h} + - W_a \Delta_h\|\,\|x_{a+\Delta h}\| + \|\Delta h\|^{-1}\|-W_a\Delta h(x_{a+\Delta h}-x_a)\| + \\
& +|\Delta h|^{-1}\|(T_a G_{a+\Delta h} x_{a+\Delta h} - T_a G_a x_a - C\Delta h - D(x_{a+\Delta h}-x_a)\| + \\
& +|\Delta h|^{-1}\|P_{a+\Delta h} - P_a - B_a\Delta h\|\,\|x_{a+\Delta h}\| + |\Delta h|^{-1}\|B_a\Delta h(x_{a+\Delta h}-x_a)\|\,.
\end{aligned}$$

By Assumptions Z_1—Z_5 it follows that

$$|\Delta h|^{-1}\|x_{a+\Delta h}-x_a+(W_a\Delta h)x_a - C\Delta h - D(x_{a+\Delta h}-x_a) - (B_a\Delta h)x_a\| \xrightarrow[\Delta h \to 0]{} 0$$

$$\begin{aligned}
& |\Delta h|^{-1}\|(I-D)(x_{a+\Delta h}-x_a) + (W_a\Delta h)x_a - C\Delta h - (B_a\Delta h)x_a\| = \\
& = \|(I-D)(|\Delta h|^{-1}(x_{a+\Delta h}-x_a+(I-D)^{-1}(W_a\Delta h)x_a - C\Delta h - (B_a\Delta h)x_a))\|\,.
\end{aligned}$$

Hence and by Remark 3 we get

$$|\Delta h|^{-1}||x_{a+\Delta h}-x_a+(I-D)^{-1}((W_a\Delta h)x_a-C\Delta h)-(B_a\Delta h)x_a)|| \to 0, \quad \text{as } h \to 0 .$$

PROPOSITION 1. 1° *Under Assumption* Z_3 *the following two statements are equivalent*:

(a) $\dim N_a < \infty$ *and the mapping* $\mathscr{U} \ni h \to P_h \in B(X)$ *is continuous at* a.

(b) *There exists an open neighborhood* V_a *of* a *and a positive integer* k *such that* $\dim N_h = k$ *for* $h \in V_a$.

2° *If* Z_3 *together with either (a) or (b) hold then Assumption* Z_4 *holds too.*

Proof. (a) ⇒ (b). Since the mapping $\mathscr{U} \ni h \to P_h \in B(X)$ is continuous at a, there exists an open neighborhood $\tilde{U}_a$ of a such that $||P_h-P_a|| < 1$ for $h \in \tilde{U}_a$.

If P_a is the zero projection, then $N_a = \{0\}$ and $||P_h-P_a|| = ||P_h|| < 1$. Thus $P_h(x) = 0$ for all $x \in X$, $h \in \tilde{U}_a$. Hence $N_h = \{0\}$ for $h \in \tilde{U}_a$.

If P_a is a non-zero projection then it follows from Lemma 12.15 in [1], p. 197 that $\dim N_h = \dim N_a$, for all $h \in \tilde{U}_a$.

(b) ⇒ (a). Following Theorem 2 of [7] there exists an open neighbourhood $V_a \subset \tilde{U}_a$ and a base $e_1(h), \ldots, e_k(h)$ of N_h, for $h \in V_a$, such that the mappings

$$V_a \ni h \to e_j(h) \in N_h, \quad j = 1, \ldots, k$$

are differentiable at a. We can assume that the base is orthonormal for all $h \in V_a$. Then

$$P_h \sum_{j=1}^{k} <\cdot, e_j(h)> e_j(h) \qquad \text{for } h \in V_a,$$

where $<\cdot, \cdot>$ is the scalar product in X.

To prove differentiability let us observe that the mapping

$$V_a \ni h \to \; <\cdot, e_j(h)> e_j(h) \in B(X)$$

can be presented as the composition $\varphi \circ \psi$, where φ, ψ are given by

$$\varphi\colon\ X \times X \ni (v, w) \to \; <\cdot, v> w \in B(x)$$

$$\psi\colon\ V_a \ni h \to (e_j(h), e_j(h)) \in X \times X .$$

Since φ is of class C^∞ and ψ is differentiable at a, the mapping $\varphi \circ \psi$ is differentiable at a.

Let us observe that the assumption of contractivity of $T_h G_h$ has been used only to prove that the mapping $h \to x_h$ is bounded in a neighbourhood of a (see proof of Th. 1). Now, we present another approach. Let x_h, for $h \in U$ be the same as in Theorem 1.

PROPOSITION 2. *If* 1° $\dim N_h = k < \infty$ *for* $h \in \mathscr{U}$, 2° *there exists* $\delta \in (0, 1)$ *and an open neighbourhood* V *of* a *such that*

$$||T_a G_w x - T_a G_w y|| \leqslant \delta ||x-y||$$

for all $w \in V$, $x, y \in D$,
3° *the mapping*

$$\mathscr{U} \ni h \to T_a G_h(0) \in X$$

is bounded,

4° *the mapping*

$$\mathscr{U} \ni h \to [T_a A_h] \in B(X)$$

is continuous at a,
then the mapping

$$h \to x_h \in X$$

is bounded.

Proof. Since $T_a A_a = I - P_a$, we have $[T_a A_h] + P_a = I + [T_a(A_h - A_a)]$. Then using Assumption 4° we see that there exists an open neighbourhood U of a such that

$$\|[T_a(A_h - A_a)]\| < \frac{1-\delta}{2}.$$

Thus, the mapping $[T_a A_h] + P_a$ is bijective and $V_h = ([T_a A_h] + P_a)^{-1}$ is a bounded linear mapping and

$$\|V_h\| \leqslant \frac{2}{1+\delta}.$$

Getting back to the definition of x_h and V_h, we have

$$x_h = V_h T_a G_h x_h + V_h P_a x_h.$$

Since x_h is orthogonal to $R(V_h P_a)$ (cf. [5]), we have

$$|x_h|^2 = (V_h T_a G_h x_h, x_h).$$

Thus, by the Schwarz and the triangle inequality

$$|x_h| \leqslant \|V_h\| \, |T_a G_h x_h - T_a G_h(0)| + \|V_h\| \, |T_a G_h(0)|.$$

Therefore, there exists a positive constant C_1 such that

$$|x_h| \leqslant \frac{2\delta}{1+\delta} |x_h| + C_1.$$

Hence

$$|x_h| \leqslant \frac{1+\delta}{1-\delta} C_1.$$

Remark. To obtain k-times differentiability of $h \to x_h$ it suffices to assume additionally that

1. $\operatorname{Ker} A_h = \{0\}$ for $h \in \mathscr{U}$,
2. $h \to T_a A_h$ is k-times differentiable at a,
3. $(h, x) \to T_a G_h(x)$ is k-times differentiable at the point (a, x_a).

Indeed, differentiating both sides of the equation

$$[T_a A_h] x_h = [T_a G_h] x_h$$

with respect to h, we get a new equation for x_h' of the form

(*) $$[T_a A_h] x_h' = D x_h' + f(h, x_h),$$

where D is the partial derivative of $T_a G$ discussed in the proof of Theorem 1. It is easy to see that $(*)$ satisfies all the assumptions of Theorem 1. Hence $h \to x_h'$ is differentiable at a. Now, the continuation of inductive reasoning is clear.

Example. Let G be a bounded domain in $\boldsymbol{R}^k$ with the boundary ∂G of class C^∞. Let

$$A_h = A_h(x, D) = \sum_{|\mu| < m} a_\mu(x, h) D^\mu$$

be a properly elliptic operator of order m having coefficients a_μ of class C^∞ in $\mathscr{U} \times \bar{G}$, where $\mathscr{U}$ is an open subset of $\boldsymbol{R}^n$. Let

$$B_j = B_j(x, D) = \sum_{|\mu| \leqslant m_j} b_{j\mu}(x) D^\mu, \qquad 1 \leqslant j \leqslant \frac{m}{2}$$

be a set of $\frac{m}{2}$ boundary operators with coefficients $b_{j\mu}$ of class C^∞ in $\bar{G}$ such that B_j, $j = 1, \ldots, \frac{m}{2}$ cover A_h, for all $h \in \mathscr{U}$ (for detailed definitions see e.g. [3]).

Let $X = Y = L^2(G)$, $D = \left\{v \in H^m(G)\colon B_j v = 0,\ j = 1, \ldots, \frac{m}{2}\right\}$ where $H^m(G)$ is the m-th Sobolev space. Then it follows from the theory of elliptic operators that

1° $N_h = \{v \in D\colon A_h v = 0\}$ is a closed subspace of X
2° $R(A_h)$ is a closed subspace of Y
3° $A_h\colon X \to Y$ is a closed operator.

Therefore, there exists a bounded operator $T_h\colon Y \to X_h = N_h^\perp \subset X$ such that

$$T_h A_h x = x - P_h x \qquad \text{for } x \in D,\ h \in \mathscr{U}.$$

Suppose that $\dim N_h = k < \infty$ for every $h \in \mathscr{U}$. M. Schechter proved in [5] that Assumption Z_3 holds. Therefore, Theorem 2 proved in [7] states that there exists an open neighbourhood $\mathscr{U}_a$ of a and a base $e_1(h), \ldots, e_k(h)$ of N_h such that the mappings

$$\mathscr{U}_a \ni h \to e_j(h) \in X, \quad j = 1, \ldots, k$$

are differentiable at a. Hence, by Proposition 1 the mapping

$$\mathscr{U} \ni h \to P_h \in B(X)$$

is differentiable at a. Hence, we can use Theorem 1 to study differentiability with respect to parameter h the solution $u_h \in N_h^\perp$ of an elliptic non-linear equation

$$\text{(i)} \qquad A_h u = G_h u$$

with a non-linear right hand side G_h of (i) which satisfies assumptions of Theorem 1.

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