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Evolution equations with parameter

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Abstract. The purpose of this paper is to give theorems on continuity, differentiability and C^1-class with respect to (h, t) of the solution of the evolution equation $\frac{du_h(t)}{dt} = A_h u_h(t)+f_h(t)$, $u_h(0) = u_h^0$ with parameter $h \in \Omega \subset \boldsymbol{R}^n$.

1. Preliminaries. Assuming that X, Y are Banach spaces we let $B(X, Y)$ be the Banach space of all linear bounded operators and $\mathscr{C}(X, Y)$ be the vector space of all linear, closed operators from X into Y. If $A\colon X \to Y$ is a linear operator then $D(A)$, $N(A)$, $R(A)$, $\bar{A}$, $P(A)$ will denote the domain, kernel, range, closure and the resolvent set of A, respectively.

The following simple Lemma will be useful for us

LEMMA 1. *Let X, Y, Z be Banach spaces, $T \in B(Z, X)$ and $A \in \mathscr{C}(X, Y)$. If $R(T) \subset D(A)$ then $AT \in B(Z, Y)$.*

Proof. It follows immediately from the Banach closed graph theorem.

Let Ω be an open subset of $\boldsymbol{R}^n$. Let $(A_h)_{h\in\Omega}$ be a family of linear operators $A_h\colon X \to Y$ with a domain $D_h = D(A_h)$, for $h \in \Omega$.

Definition 1 (cf. [2], p. 460). We call the family $(A_h)_{h\in\Omega}$ R-continuous at a point $h_0 \in \Omega$ if there exist a Banach space Z and a family $T_h \in B(Z, X)$, $h \in \Omega$ such that

(a) $R(T_h) = D_h$ and the mapping $Z \ni z \to T_h(z) \in D_h$ is bijective for all $h \in \Omega$, and
(b) the mappings

$\Omega \ni h \to T_h \in B(Z, X)$ and $\Omega \ni h \to V_h = A_h T_h \in B(Z, Y)$ are continuous at the point h_0.

The continuity in Ω is defined by the natural manner as continuity at every point of Ω.

Remark. Definition 1 is a formal adoption of the Rellich's [3] and Kato's [2] definition of holomorphic families of operators. It is easy to see that for families of bounded operators R-continuity and coincide.

We shall use the following simple Lemma.

LEMMA 2. *Let $A_h \in \mathscr{C}(X, Y)$ for $h \in \Omega$ and suppose that A_h: $D_h \to Y$ is bijective for all $h \in \Omega$. Then the mapping*

$$\Omega \ni h \to A_h \in \mathscr{C}(X, Y)$$

is R-continuous at a point $h_0 \in \Omega$ if and only if the mapping

$$\Omega \ni h \to A_h^{-1} \in B(Y, X)$$

is continuous at h_0.

Proof. To prove the sufficient condition it suffices to take $Z = Y$ and $T_h = A_h^{-1}$ for $h \in \Omega$.

Now let Z, T_h. V_h be the same as in Definition 1. Since $A_h T_h = V_h$ and, by assumptions, all operators are invertible we have

$$A_h^{-1} = T_h V_h^{-1} \quad \text{for } h \in \Omega .$$

By Banach Theorem V_h^{-1} is bounded for $h \in \Omega$ and since $h \to V_h$ is continuous, the mapping $h \to V_h^{-1}$ is also continuous. Hence the mapping $h \to A_h^{-1}$ is continuous.

COROLLARY 1. *Suppose that $X = Y$ and $\lambda \in P(A_h)$ for all $h \in \Omega$. Then the mapping*

$$\Omega \ni h \to A_h \in \mathscr{C}(X) = \mathscr{C}(X, X)$$

is R-continuous at a point $h_0 \in \Omega$ if and only if the mapping

$$\Omega \ni h \to (A_h - \lambda I)^{-1} \in B(X)$$

is continuous at h_0.

2. Continuity with respect to parameter. We start with a preparatory material.

LEMMA 3. *If the mapping*

$$\Omega \ni h \to A_h \in \mathscr{C}(X) \tag{1}$$

is R-continuous at a point $h_0 \in \Omega$, then for every compact set $K \subset P(A_{h0})$ there exists $\delta > 0$ such that $K \subset P(A_h)$, whenever $|h - h_0| < \delta$, where $|\cdot|$ is a norm in $\mathbf{R}^n$.

Proof. It is a simple consequence of Theorems 2.29 and 3.1 of Kato [2].

As a simple consequence of Lemma 3 we obtain.

LEMMA 4. *If the mapping* (1) *is R-continuous then the set*

$$U = \{(\lambda, h) \in \mathbf{C} \times \Omega \mid \lambda \in P(A_h)\}$$

is open in $\mathbf{C} \times \Omega$.

LEMMA 5. *If the mapping* (1) *is R-continuous at the point $h_0 \in \Omega$, then the mapping*

$$U \ni (\lambda, h) \to R(\lambda, A_h) = (A_h - \lambda I)^{-1} \in B(X) = B(X, X)$$

is continuous at $(\lambda, h_0) \in U$.

Proof. The mapping

$$U \ni (\lambda, h) \to (A_h - \lambda I) \in \mathscr{C}(X)$$

is R-continuous at (λ, h_0). Thus the Lemma 5 is proved remembering Lemma 2.

Setting

$$G(M, \beta) = \{A \in \mathscr{C}(X) \colon \overline{D(A)} = X, \ (\beta, +\infty) \subset P(A)$$

and

$$\|(A-\xi)^{-k}\| \leqslant M(\xi-\beta)^{-k} \quad \text{for } \xi > \beta, \ k = 1, 2, \ldots\}$$

we shall prove the following proposition.

PROPOSITION 1. *Let K be a compact subset of X. If $A_h \in G(M, \beta)$ for $h \in \Omega$, $h_0 \in \Omega$ and the family $(A_h)_{h\in\Omega}$ is R-continuous in h_0, then for every $x \in X$*

$$\lim_{h \to h_0} e^{tA_h} x = e^{tA_{h_0}} x$$

uniformly in $[0, T]xK$.

Proof. Let us put

$$\Phi(t, h) = e^{tA_h} - e^{tA_{h_0}}$$

Following Corollary 1, if $\zeta \in P(A_h)$ then $(A_h - \zeta)^1 \to (A_{h_0} - \zeta)^{-1}$ when $h \to h_0$. Hence (see [2], Th. 2.16) $\Phi(t, h)x \to 0$ for $x \in X$ whenever $h \to h_0$, uniformly with respect to $t \in [0, T]$. By the assumption $A_h \in G(M, \beta)$ for $h \in \Omega$ we have inequality

$$\|\Phi(t, h)x\| \leqslant 2Me^{\beta T}\|x\| \quad \text{for } t \in [0, T], \ x \in X.$$

Then see e.g. [4], Th. 2.5, p. 111) the family $\Phi(t, h)$ is equi-continuous family of operators, so for every $x_0 \in X$ and $\varepsilon > 0$ there exists $\delta = \delta(x_0) > 0$ such that

$$\Phi(t, h)\big(B(x_0, \delta)\big) \subset B\left(\Phi(t, h)x_0; \frac{\varepsilon}{2}\right),$$

where $B(x_0, \delta) \subset X$ is the ball of radius δ centered at x_0. Since $K \subset X$ is a compact set, we can choose a finite set $x_1, \ldots, x_l$ of K such that

$$K \subset \sum_{j=1}^{l} B\big(x_j, \delta(x_j)\big).$$

Since $\Phi(t, h)x \to 0$ for all $x \in X$ when $h \to h_0$ uniformly with respect to $t \in [0, T]$, there exists $\mu > 0$ such that

$$\|\Phi(t, h)x_s\| < \frac{\varepsilon}{2} \quad \text{for } s = 1, 2, \ldots, l$$

whenever $|h-h_0|<\mu$ and $t\in[0,T]$. Thus, by inclusion (*)

$$\Phi(t,h)(K)\subset B(0,\varepsilon)$$

whenever $|h-h_0|<\mu$ and $t\in[0,T]$.

Considering the family of evolution equations

$$\frac{du_h}{dt}=A_h u_h+f_h \tag{2}$$

with the initial condition

$$u_h(0)=u_h^9 \tag{3}$$

we can prove the following.

THEOREM 1. *Suppose that $A_h\in G(M,\beta)$, $D(A_h)=D$, $u_h^0\in D$ for $h\in\Omega$ the family $(A_h)_{h\in\Omega}$ is R-continuous at a point $h_0\in\Omega$, the mapping $\Omega\ni h\to u_h^0\in X$ is continuous, $f\colon \Omega x[0,T]\to X$ is continuous and $f_h=f(h,\cdot)\colon[0,T]\to X$ is C^1 class for $h\in\Omega$. Then for every $h\in\Omega$ there exists exactly one solution u_h of the problem* (2)—(3) *and*

$$\lim_{h->h_0} u_h(t)=u_{h_0}(t)$$

uniformly with respect to $t\in[0,T]$.

Proof. It is well-known (see e.g. [2]) that with our assumptions, the solution of the problem (2)—(3) has the form

$$u_h(t)=e^{tA_h}u_n^0+\int_0^t e^{(t-s)A_h}f_h(s)\,ds\,. \tag{4}$$

Thus, by standard calculation we obtain

$$\begin{aligned}u_h(t)-u_{h_0}(t)&=(e^{tA_h}-e^{tA_{h_0}})u_h^0+e^{tA_{h_0}}(u_h^0-u_{h_0}^0)+\\&+\int_0^t[e^{(t-s)A_h}-e^{(t-s)A_{h_0}}]f_h(s)\,ds+\int_0^t e^{(t-s)A_{h_0}}[f_h(s)-f_{h_0}(s)]\,ds\,.\end{aligned} \tag{5}$$

Let K_1 be a compact neighbourhood of the point h_0. Since the mapping $h\to u_h^0$ is continuous, the set $K=\{u_h^0\colon h\in K_1\}$ is a compact subset X and by Proposition 1

$$(e^{tA_h}-e^{tA_{h_0}})u_h^0\underset{h\to h_0}{\to}0$$

uniformly in $[0,T]$.

Since there exists $M>0$ such that $\|e^{tA_{h_0}}\|\leqslant M$, for $t\in[0,T]$, so

$$e^{tA_{h_0}}(u_h^0-u_{h_0}^0)\underset{h\to h_0}{\to}0$$

uniformly in $[0,T]$.

Thus

$$\int_0^t[e^{(t-s)A_h}-e^{(t-s)A_{h_0}}]f_h(s)\,ds\underset{h\to h_0}{\to}0$$

uniformly in $[0,T]$.

Since f is continuous,

$$\int_0^t e^{(t-s)A_{h_0}}(f_h(s)-f_{h_0}(s))\,ds \underset{h\to h_0}{\to} 0$$

uniformly with respect to $t\in[0,T]$. This proves the Theorem.

COROLLARY. *Suppose that all assumption of Theorem* 1 *are satisfied. Then following* [2] *Theorem* 1.19, *the solution*

$$u\colon \Omega x[0,T]\to X$$

of the problem (2)—(3) *is continuous at every point* (h_0, t), $t\in[0,T]$.

3. Differentiability with respect to parametr. Let D be a normed vector space over R such that there exist a Banach space Z and a linear, bounded, bijective mapping $T\colon Z\to D$. Setting $sB(D, Y) = \{A\colon D\to Y\colon A$ is linear and $AT\in B(Z, Y)\}$ we see that $sB(D, Y)$ is independent of (Z, T). Indeed if $T_1\colon Z_1\to D$ is another linear, bounded bijective mapping of a Banach space Z_1 onto D then

$$AT_1 = (AT)(T^{-1}T_1) \text{ and, by Lemma 1 } AT_1\in B(Z_1, Y).$$

Definition. Let Ω be an open subset of $\boldsymbol{R}$. A function $\Omega\ni h\to A_h\in sB(X, Y)$ is said to be differentiable (continuous) at a point $h_0\in\Omega$ if there exist a Banach space Z and a linear, bounded, bijective mapping $T\colon Z\to D$ such that the mapping $\Omega\ni h\to A_h T\in B(Z, D)$ is differentiable (continuous) in the Frechet sense. In this case we put

$$A_{h_0} = \left(\frac{d}{dh}(A_hT)\Big|_{h=h_0}\right)T^{-1}.$$

Since

$$A_hT_1 = (A_hT)(T^{-1}T_1)$$

we see that the definition of differentiability is independed on choosing of Z, T. Moreover,

$$A'_{h_0}u = \lim_{h\to h_0}\frac{A_hT-A_{h_0}T}{h-h_0}T^{-1}u = \lim_{h\to h_0}\frac{A_h-A_{h_0}}{h-h_0}u$$

for $u\in D$.

Thus $A'_{h_0}\in sB(D, Y)$ and it is independent of Z, T.

Now, if Ω is an open subset of $\boldsymbol{R}^n$ and $h_0\in\Omega$, we can define partial derivatives $\frac{\partial A_h}{\partial h_j}$, $j=1,\dots,n$ and the C^k-class of the mapping $h\to A_h\in sB(X, Y)$ in the standard manner. Keeping notations of previous sections we prove

LEMMA 6. *If* $A\in G(M,\beta)$, $f\colon [0,\infty)\to X$ *is of* C^2*-class* $Au^0+f(0)\in D(A)$ *and* u *is the solution of the problem*

$$\frac{du}{dt} = Au+f \quad \text{with } u(0) = u^0,$$

then, u *is of* C^2*-class in* $[0,\infty)$.

Proof. Putting $v = u - u^0$ we see that

$$\frac{dv}{dt} = Av + (Au^0 + f) \quad \text{and} \quad v(0) = 0$$

so, by virtue of (4)

$$v(t) = \int_0^t T(s)(Au^0 + f(t-s))\,ds,$$

where $T(s) = e^{sA}$. Therefore

$$\frac{dv}{dt} = \int_0^t T(s) f'(t-s)\,ds + T(t)(Au^0 + f(0)).$$

Thus $w = \frac{dv}{dt} = \frac{du}{dt}$ is the solution of the problem

$$\frac{dw}{dt} = Aw + \frac{df}{dt}, \quad w(0) = Au^0 + f(0) \tag{6}$$

and by Th. 1.19 of [2] we conclude that u is of C^2-class.

Now, we return to the family of operators $(A_h)_{h\in\Omega}$ which satisfies assumption of Theorem 1. In view of the fact $w_h = du_h/dt$ is the solution of problem (6) for $h \in \Omega$ we have the following.

LEMMA 7. *Suppose that all assumptions of Theorem* 1 *are satisfied at every point* $h_0 \in \Omega$. *If in addition* $A_h u_h^0 + f_h(0) \in D$ *for* $h \in \Omega$, $f_h = f(h, \cdot)\colon [0, T] \to X$ *is of class* C^2, $\frac{df_h}{dt}\colon \Omega x [0, T] \to X$ *is continuous and* $\Omega \ni h \to A_h u_h^0$ *is continuous, then the mapping*

$$\Omega x [0, T] \ni (h, t) \to \frac{du_h}{dt}(t) \in X$$

is continuous.

We are now able to prove

THEOREM 2. *Let* Ω *be an open set in* $\mathbf{R}$, $A_h \in G(M, \beta)$, $D(A_h) = D$, *and* $u_h^0 \in D$ *for* $h \in \Omega$. If

1° *the mapping* $\Omega \ni h \to A_h$ *is continuous in* Ω *and differentiable at* $h_0 \in \Omega$,

2° *the mapping* $\Omega \ni h \to u_h^0$ *is continuous in* Ω *and differentiable at* $h_0 \in \Omega$,

3° $f_h\colon [0, T] \to f_h(t)$ *is of* C^2-*class for* $h \in \Omega$,

4° $\Omega x [0, T] \ni (h, t) \to \frac{\partial f}{\partial h}(h, t)$ *is continuous,*

5° $A_h u_h^0 + f_h(0) \in D$ *for* $h \in \Omega$

and

6° $\Omega \ni h \to A_h u_h^0 + f_h(0)$ is continuous then for every h there exists exactly one solution $u_h(t) = u(h, t)$ of the problem

$$\frac{du_h}{dt} = A_h u_h + f_h, \quad u_h(0) = u_h^0$$

which is of C^1-class with respect to t and differentiable with respect to h at the point h_0. Moreover

$$\lim_{h \to h_0} \frac{u_h(t) - u_{h_0}(t)}{h - h_0} = u'_{h_0}(t),$$

uniformly with respect to $t \in [0, T]$ and u'_{h_0} is the solution of the problem

$$\frac{d}{dt} u'_{h_0} = A_{h_0} u'_{h_0} + A'_{h_0} u_{h_0} + f'_{h_0} u'_{h_0}(0) = (u^0_{h_0}),$$

where the sign "′" denotes differentation with respect to h.

Proof. Existence and uniqueness follow similarly as in Theorem 1.

Let $h \in \Omega$ be different from h_0. We first observe that

$$(7) \quad \frac{d}{dt}\left(\frac{u_h - u_{h_0}}{h - h_0}\right) = A_h \left(\frac{u_h - u_{h_0}}{h - h_0}\right) + \frac{A_h - A_{h_0}}{h - h_0} u_{h_0} + \frac{f_h - f_{h_0}}{h - h_0}$$

and

$$(8) \quad \frac{u_h(0) - u_{h_0}(0)}{h - h_0} = \frac{u_h^0 - u_{h_0}^0}{h - h_0}.$$

If we take

$$F_h = \begin{cases} \dfrac{A_h - A_{h_0}}{h - h_0} + \dfrac{f_h - f_{h_0}}{h - h_0} & \text{for } h \neq h_0 \\ A'_{h_0} u_{h_0} + f'_{h_0} & \text{for } h = h_0 \end{cases}$$

$$v_h^0 = \begin{cases} \dfrac{u_h^0 - u_{h_0}^0}{h - h_0} & \text{for } h \neq h_0 \\ (u_{h_0}^0)' & \text{for } h = h_0 \end{cases}$$

and

$$v_h = \frac{u_h - u_{h_0}}{h - h_0} \quad \text{for } h \neq h_0$$

then we see that v_h, $h \neq h_0$, is the solution of the problem

$$(9) \quad \frac{dv_h}{dt} = A_h v_h + F_h$$

$$(10) \quad v_h(0) = v_h^0.$$

Therefore the Theorem 2 is proved if we can show that Theorem 1 can be applied. Since the family $(A_h)_{h \in \Omega}$ and the mapping $\Omega \ni h \to v_h^0 \in X$ satisfy clearly all the assumptions of Theorem 1, we have to prove only that the mapping $\Omega \ni h \to F_h$ fulfils then as well. Taking $\lambda \in P(A_{h_0})$ and $T = (A_{h_0} - \lambda I)^{-1}$ we have

$$\frac{A_h - A_{h_0}}{h - h_0} u_{h_0}(t) = \left(\frac{A_h - A_{h_0}}{h - h_0} T\right) T^{-1} u_{h_0}(t).$$

Then, by Lemma 6, $T^{-1} u_{h_0}$ is of C^1-class in $[0, T]$ and since the mapping $h \to A_h$ is continuous and differentiable at h_0, we conclude that the mapping

$$\Omega x [0, T] \ni (h, t) \to F_h(t) \in X$$

is continuous. Thus, as we have remarked above, this proves our theorem.

4. C^1-class. Now, we formulate the main theorem of this paper

THEOREM 3. *Let Ω be an open subset of R^n. Suppose that $A_h \in G(M, \beta)$, $D(A_h) = D$, $u_h^0 \in D$ for $h \in \Omega$, the mapping $\Omega \ni h \to u_h^0 \in X$ is of C^1-class, $f: \Omega x [0, T] \to X$ is of C^1-class, $f_h = f(h, \cdot)$: $[0, T] \to X$ is of C^2-class, $\Omega \ni h \to A_h$ is of C^1-class, $A_h u_h^0 + f_h(0) \in D$ for $h \in \Omega$ and the mapping $\Omega \ni h \to A_h h_h^0 + f_h(0)$ is continuous. Then there exists exactly one solution $u_h(t) = u(h, t)$ of the problem (2)—(3) and it is of C^1-class in $\Omega x [0, T]$.*

Proof. Existence, uniqueness and continuity of solution u follows from Theorem 1.

Following Lemma 7 the solution u has continuous partial derivative $\frac{\partial u}{\partial t}$. Fixing $j \in \{1, \ldots, n\}$ we denote by $u_h', f_h', A_h', (u_h^0)'$ the partial derivative $\frac{\partial}{\partial h_j}$ of u, f, A_h, u_h^0, respectively. Existence of u_h' follows from Theorem 2. Since u_h' is the solution of the problem

$$\frac{d}{dt} u_h = A_h u_h' + F_h, \tag{11}$$

$$u_h'(0) = (u_h^0)' \tag{12}$$

with $F_h = A_h' u_h + f_h'$, we have to prove only that the mappings

$$\Omega x [0, T] \ni (h, t) \to A_h' u_h(t) \in X \tag{13}$$

and

$$[0, T] \ni t \to \frac{d}{dt} A_h' u_h(t) \in X \quad \text{for } h \in \Omega \tag{14}$$

are continuous and apply Theorem 1.

Let $T_h = (A_h - \lambda I)^{-1}$ for $h \in \Omega$ with a fixed $\lambda > \beta$. Then we have

$$A_h' u_h = (A_h' T_{h_0})(T_{h_0}^{-1} T_h)(T_h^{-1} u_h).$$

Since $h \to A_h$ is of C^1-class, the mapping

$$\Omega x [0, T] \ni (h, t) \to A_h' T_h \in B(x)$$

is continuous and of C^{∞}-class with respect to t. Thus, again by the some assumption, we see that the mapping

$$\Omega x[0, T] \ni (h, t) \to (T_h^{-1} T_{h_0})^{-1} = T_{h_0}^{-1} T_h \in B(X)$$

is of C^1-class with respect to t. Then, by the above, we conclude that the mappings (13) and (14) are continuous.

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