UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXVII

1988

# A theorem of Bernstein's type for linear projections

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**Introduction.** Let $X$ be a normed linear space over $\boldsymbol{R}$ or $\boldsymbol{C}$ and let $Y$ be its closed linear subspace. A linear operator $P\colon X \to Y$ is called a projection if $P$ is continuous and $Py = y$ for each $y \in Y$. The set of all projections $P\colon X \to Y$ is denoted by $\mathscr{P}(X, Y)$. For the basic properties of projections in normed linear spaces see e.g. [1], [2], [3].

For a Banach space $X$ the following Bernstein theorem is well known:

THEOREM 1. *Let $V_1 \subset V_2 \subset \dots$ be a nested sequence of finite-dimensional distinct vector subspaces of a Banach space $X$ and let $M_i\colon X \to V_i$ $(i = 1, 2, \dots)$ be the operator of the best approximation, i.e.* $\|x - M_i x\| = \operatorname{dist}(x, V_i)$ $(i = 1, 2, \dots)$. *For any decreasing sequence $\varepsilon_1 \geqslant \varepsilon_2 \geqslant \dots$ with* $\lim \varepsilon_i = 0$, *there exists a point $x \in X$ such that $\|x - M_i x\| = \varepsilon_i$ for $i = 1, 2, \dots$*

The proof of this theorem (in a more general statement) can be found in [4].

In numerous problems of the approximation theory we have an interest to replace the operators $M_i$ (which are in general set-valued) by linear projections. The quality of the approximations obtained from a projection $P \in \mathscr{P}(X, Y)$ is controlled by the inequality

$$\|x - Px\| \leqslant \|I - P\| \operatorname{dist}(x, Y) \quad (x \in X).$$

In this note we consider the problem of whether the sequence of operators $(M_i)_{i=1}^{\infty}$ in Theorem 1 may be replaced by a sequence of operators $(P_i)_{i=1}^{\infty}$ such that $P_i \in \mathscr{P}(X, V_i)$ $(i = 1, 2, \dots)$. We shall prove that such a theorem still holds under an additional assumption regarding the sequence $(\varepsilon_i)_{i=1}^{\infty}$. Next we shall point out that this additional assumption may be omitted in the case of some sequences of projections. Finally some examples will be presented.

## 1. The main theorem

THEOREM 2. *Let* $V_1 \subset V_2 \subset ..$ *be a nested sequence of distinct finite-dimensional vector subspaces of a Banach space* $X$ *and* $(P_i)_{i=1}^{\infty}$ *be a sequence of projections such that* $P_i \in \mathscr{P}(X, V_i)$ $(i = 1, 2, \ldots)$. *Let*

$$\|I-P_i\|_1 := \sup\{\|x-P_ix\|:\ x \in \bigcup_{i=1}^{\infty} V_i,\ \|x\| \leqslant 1\}\ (i = 1, 2, \ldots).$$

*Then for every sequence* $(\varepsilon_i)_{i=1}^{\infty}$ *of nonnegative real numbers such that*

$$\varepsilon_i \geqslant \|I-P_i\|_1 \cdot \varepsilon_{i+1} \qquad i = 1, 2, \ldots, \tag{1}$$

*and*

$$\lim \varepsilon_i = 0 \tag{2}$$

*there exists an element* $x \in X$ *such that* $\|x-P_ix\| = \varepsilon_i$ *for* $i = 1, 2, \ldots$

Proof. Let $F_n = \{x \in X:\ \|x-P_ix\| = \varepsilon_i,\ i = 1, 2, \ldots, n\}$. We shall prove that $F_n \neq \phi$ for each $n \in N$. Let us fix $n \in N$. We construct a sequence of vectors $u_n, \ldots, u_1$ such that

1° $u_i \in V_{n+1},\ i = 1, 2, \ldots, n,$
2° $\|u_i - P_j u_i\| = \varepsilon_j,\ i = 1, 2 \ldots, n,\ i \leqslant j \leqslant n,$
3° $P_i u_i = 0,\ i = 1, 2, \ldots, n.$

Let us choose a vector $u_n \in V_{n+1}$ such that $P_n u_n = 0$ and $\|u_n\| = \varepsilon_n$. It is easily seen that $u_n$ fulfils conditions $1^0$—$3^0$. Let us assume that vectors $u_n, \ldots, u_k$ $(k \geqslant 2)$ with conditions $1^0$—$3^0$ has already been constructed. Now we shall find $u_{k-1}$. We note that the inequalities

$$\|u_k - P_{k-1}u_k\| \leqslant \|I-P_{k-1}\|_1 \|u_k\| \leqslant \varepsilon_{k-1}$$

follow from the assumption. So we can choose an element $v_k \in V_k$ such that

$$\|u_k - P_{k-1}u_k + v_k\| = \varepsilon_{k-1} \text{ and } P_{k-1}v_k = 0.$$

Let us set $u_{k-1} := u_k - P_{k-1}u_k + v_k$. We observe that $u_{k-1} \in V_{n+1}$ and

$$P_{k-1}u_{k-1} = P_{k-1}(u_k - P_{k-1}u_k + v_k) = P_{k-1}(u_k - P_{k-1}u_k) + P_{k-1}v_k = 0.$$

Consequently

$$\|u_{k-1} - P_{k-1}u_{k-1}\| = \|u_{k-1}\| = \varepsilon_{k-1}.$$

By the induction hypothesis we have, for $k \leqslant j \leqslant n$,

$$\|u_{k-1} - P_j u_{k-1}\| = \|u_k - P_{k-1}u_k + v_k - P_j(u_k - P_{k-1}u_k + v_k)\| = \|u_k - P_j u_k\| = \varepsilon_j.$$

So the sequence $\{u_n\}$ is already constructed. It is easily seen that $u_1 \in F_n$ and $\|u_1\| = \varepsilon_1$. Consequently, for every $n \in N$ there exists $x_n \in F_n$ such that $\|x_n\| = \varepsilon_1$. The final part of the proof goes on the same lines as in Bernstein's theorem. Let $v_l^n$ be an element of $V_l$ such that $\|x_n - v_l^n\| = \operatorname{dist}(x_n, V_l)$ $(n, l = 1, 2, \ldots)$. By the assumptions, applying the diagonal argument we can choose a sequence $(k_n)_{n=1}^{\infty}$ such that $v_l^{k_n} \to v_l \in V_l$ for every $l \in N$. We note that for $k_n \geqslant l$

$$||x_{k_n} - v_l^{k_n}|| \leqslant ||x_{k_n} - P_l x_{k_n}|| = \varepsilon_l .$$

Thus

$$||x_{k_n} - x_{k_m}|| \leqslant ||x_{k_n} - v_l^{k_n}|| + ||x_{k_m} - v_l^{k_m}|| + ||v_l^{k_n} - v_l|| + ||v_l^{k_m} - v_l|| \leqslant 4\varepsilon_l$$

for $m, n \geqslant n_0(l)$, where $n_0(l)$, is such that $||v_l^{k_n} - v_l|| \leqslant \varepsilon_l$ and $||x_{k_n} - v_l^{k_n}|| \leqslant \varepsilon_l$ for $n \geqslant n_0(l)$. Since $X$ is a Banach space $x_{k_n} \to x \in X$. By the continuity of the operators $P_i$ $(i = 1, 2, ...)$ we get

$$||x - P_i x|| = \lim_{n \to \infty} ||x_{k_n} - P_i x_{k_n}|| = \varepsilon_i ,$$

which completes the proof of the theorem.

We note that each $y \in x + V_1$ also satisfies the requirements of the theorem. It is easily seen from the proof of the theorem that the assumption that $P_i \in \mathscr{P}(X, V_i)$ can be weakened. For instance it suffices to assume that the operators $P_i$ $(i = 1, 2, ...)$ satisfy the following conditions:

1. $P_i$: $X \to V_i$;
2. $P_i(\alpha x) = \alpha P_i x$ $(x \in X,\ \alpha \geqslant 0)$;
3. $P_i(x+v) = P_i x + v$ $(x \in X,\ v \in V_i)$;
4. $P_i$ is continuous;
5. If $P_i x = 0$ and $P_i v = 0$ then $P_i(x+v) = 0$ for $x \in \bigcup_{i=1}^{\infty} V_i$ and $v \in V_{i+1}$.

## 2. Remarks and examples

In the case where $X$ is a Hilbert space the operators $M_i$ defined in Theorem 1 are single-valued, linear, continuous and $||I - M_i|| = 1$ $(i = 1, 2, ...)$. Thus the above theorem is a generalization of Bernstein's theorem in this particular situation.

We note that if the operators $P_i \in \mathscr{P}(X, V_i)$ are the best approximations of the identity $I$ in $\mathscr{P}(X, V_i)$ $(i = 1, 2, ...)$ then the class of sequences $(\varepsilon_i)_{i=1}^{\infty}$ that satisfy the conditions of Theorem 2 is the broadest. In the case where $X = C_R(T)$, the space of real-valued continuous functions on a compact set $T$ with the supremum norm, a sequence of operators $(P_i)_{i=1}^{\infty}$ such that $P_i$ is minimal in $\mathscr{P}(X, V_i)$ has this property if we assume that $T$ has no isolated points (see [1] p. 274 theorem 9).

We note that the assumption (1) of Theorem 2 on the sequence $(\varepsilon_i)_{i=1}^{\infty}$ may be weakened in case of some sequences of projections $\{P_i\}$. In particular, we have

PROPOSITION 3. *Let $X$, $\{P_n\}$ and $\{V_n\}$ be as in Theorem 2. Assume furthermore that for every $n \in N$ there exists $v_n \in V_{n+1} \setminus V_n$ such that $P_i v_n = 0$ $(i = 1, 2, ..., n)$. Then for every decreasing sequence of numbers $\varepsilon_1 \geqslant \varepsilon_2 \geqslant ... \geqslant 0$ with* $\lim \varepsilon_i = 0$ *there exists a point $x \in X$ such that $||x - P_i x|| = \varepsilon_i$ for $i = 1, 2, ...$*

Proof. We shall prove like in Theorem 2 that the set $F_n \neq \emptyset$ for an arbitrary $n \in N$. Let us fix $n \in N$. We construct a sequence of vectors $u_n, \ldots, u_1$ such that

1° $u_i \in V_{n+1}$, $i = 1, 2, \ldots, n$;
2° $P_j u_i = 0$, $i = 1, 2, \ldots, n$, $1 \leqslant j \leqslant i$;
3° $\|u_j - P_j u_i\| = \varepsilon_j$, $i = 1, 2, \ldots, n$, $i \leqslant j \leqslant n$.

Let us choose $u_n \in V_{n+1}$ such that $\|u_n\| = \varepsilon_n$ and $P_j u_n = 0$ for $1 \leqslant j \leqslant n$. It is easily seen that $u_n$ fulfils conditions 1°—3°. Let us assume that vectors $u_n, \ldots, u_k$ $(k \geqslant 2)$ satisfying 1°—3° are already chosen. Now we shall find $u_{k-1}$. We note, by the induction hypothesis, that $\|u_k - P_{k-1} u_k\| = \|u_k\| \leqslant \varepsilon_{k-1}$. Hence we can choose $v_k \in V_k$ such that $P_j v_k = 0$ for $1 \leqslant j \leqslant k-1$ and $\|u_k + v_k\| = \varepsilon_{k-1}$. Let us define $u_{k-1} = u_k + v_k$. It is easily seen that the vector $u_{k-1}$ fulfils condition 1°—3°. So the sequence $u_n, \ldots, u_1$ is already constructed. We note that the vector $u_1 \in F_n$ and $\|u_1\| = \varepsilon_1$. The rest of the proof is the same as in Theorem 2.

It is easily seen from the proof of the proposition that the assumption that $P_i \in \mathscr{P}(X, V_i)$ $(i = 1, 2, \ldots)$ can be weakened like in Theorem 2 with a certain modification, viz. we must assume that if $P_i x = 0$ and $P_i y = 0$ then $P_i(x+y) = 0$ for $x, y \in \bigcup_{i=1}^{\infty} V_i$.

Corollary 4. *Let us assume that the sequence of operators* $P_n \in \mathscr{P}(X, V_n)$ $(n = 1, 2, \ldots)$ *satisfies the following condition: there exist functionals* $\varphi_n \in X^*$ $(n = 1, 2, \ldots)$ *such that* $P_n = \sum_{i=1}^{\dim V_n} \varphi_i(\cdot) \cdot y_i^n$ *for each* $n \in N$, *where* $y_i^n$ $(i = 1, \ldots, \dim V_n)$ *is a basis of* $V_n$. *Then the sequence* $(P_n)_{n=1}^{\infty}$ *fulfils the assumption of Proposition* 3.

We note that by Corollary 4 the following sequences of projections fulfil the assumptions of Proposition 3.

1) Let $X = C(T)$, the space of all complex-valued continuous functions on a compact, infinite set $T$ with the supremum norm. Let $(t_i)_{i=1}^{\infty}$ be a sequence of distinct points of $T$. Assume that $(V_n)_{n=1}^{\infty}$ is an increasing sequence of distinct finite-dimensional subspaces such that the set of evaluations $\{\hat{t}_1|_{V_n}, \ldots, \hat{t}_{\dim V_n}|_{V_n}\}$ is total over $V_n$ $(n = 1, 2, \ldots)$. Define $P_n = \sum_{i=1}^{\dim V_n} t_i(\cdot) \cdot y_i^n$, where $(y_i^n)_{i=1}^{\dim V_n}$ is a basis of the subspace $V_n$ such that $y_i^n(t_j) = \delta_{ij}$ $(i, j = 1, \ldots, \dim V_n)$. It is easily seen that $P_n \in \mathscr{P}(X, V_n)$ for $n = 1, 2, \ldots$ Then in the case where $T = [a, b] \subset \boldsymbol{R}$, $X = C_R([a, b])$. and $V_n = \pi_n$, the space of polynomials of degree $\leqslant n$ restricted to $[a, b]$, we obtain a sequence of interpolating projections such that support $P_n = \{t_1, \ldots, t_{\dim V_n}\}$.

2) Let $X = C_R([-1, 1])$ and $V_n = \pi_n$ $(n = 0, 1, \ldots)$. Let us set $S_n x = \frac{1}{2} a_0(x) T_0 + \sum_{k=1}^{n} a_k(x) T_k$, where $T_k$ is the Chebyshev polynomial od degree $k$ and

$$a_k(x) = \frac{2}{\pi} \int_{-1}^{1} f(t) T_k(t) (1-t^2)^{-\frac{1}{2}} \, dt \qquad (k = 0, 1, \ldots, n).$$