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An inverse spectral problem for linear elliptic differential operators

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1. Introduction. The inverse spectral problem that we consider in this paper originated from V. Ambarzumian's theorem [1] concerning Sturm-Liouville problem for a second-order ordinary differential equation. This theorem was extended and transferred to the case of Neumann problem for Laplace's equation on bounded regions in space $\boldsymbol{R}^2$, $\boldsymbol{R}^3$ by N. Kuznecov [7]. In the works of J. Bochenek [2]—[4] and of S. Postawa [9] the next generalizations have been done. They deal with some boundary value problems for self-adjoint elliptic operators with constant coefficients in the case of $\boldsymbol{R}^n$ with $n \geqslant 2$.

The purpose of this paper is to analyse the inverse spectral problem in a general class of self-adjoint elliptic operators, of any order, with variable coefficients. The main ideas are based on J. Bochenek's papers [2]—[4], who suggested me to study the possibility of generalization of his results in this direction. Apart from the regularity class of regions on which the boundary value problems are considered, one may say that many essential theorems of the papers [2]—[4], [9] are some special cases of Theorem 6. In this work we shall apply the main results of author's paper [8], using the notation described there in details.

I would like to thank Professor Jan Bochenek for his encouragement throughout this research.

2. Formulation of problem. We begin with short recall of some needed notations that have already been used in [8]. By Ω we denote a bounded open subset of $\boldsymbol{R}^n$ having the cone property. We consider only real function spaces: $\mathscr{D}(\Omega)$ — the set of C^∞-functions with compact supports in Ω, $L^2(\Omega)$ — the Hilbert space of square-integrable functions on Ω, with scalar product $\langle \cdot, \cdot \rangle$ and with norm $\|\cdot\|$, $H^j(\Omega)$ (where $j = 0, 1, 2, \ldots$) — the Sobolev space of order j. We denote by $\mathfrak{a}$ the linear elliptic differential operator, of even order $m = 2m_0$ (where $m_0 \in N$), of the form

$$\mathfrak{a}(x, D)u = \sum_{|p| \leqslant m} \alpha_p(x) D^p u,$$

where the coefficients α_p ($|p| \leq m$) are supposed to be (real) C^∞-functions on an open set Ω_1 such that $\Omega \subset \Omega_1$. We assume that the characteristic polynomial

$$\mathfrak{a}_0(x, \xi) = \sum_{|p|=m} \alpha_p(x)\xi^p,$$

corresponding to the principal part $\mathfrak{a}_0$ of operator $\mathfrak{a}$, satisfies the condition

$$(1) \qquad (-1)^{m_0}\mathfrak{a}_0(x, \xi) \geq 0, \quad \forall x \in \Omega, \quad \forall \xi \in \mathbf{R}^n.$$

From now on the linear operator A, with domain $D(A) \subset L^2(\Omega)$ and range $R(A) \subset L^2(\Omega)$, is assumed to be a realization of elliptic operator $\mathfrak{a}$ in the sense that $\mathscr{D}(\Omega) \subset D(A)$ and $Au = \mathfrak{a}u$ for each $u \in D(A)$, where $\mathfrak{a}u$ is understood in the distributional sense. A is also assumed to be a self-adjoint bounded from below operator with compact resolvent.

With the above notations and assumptions we shall be concerned with the spectra of operator A and of $A+V$, where V is a multiplier operator q, with $q \in L^\infty(\Omega)$ (that is, $(Vu)(x) = q(x)u(x)$, for $x \in \Omega$). It is obvious that V is a self-adjoint bounded linear operator in $L^2(\Omega)$. Let $\{\lambda_j\}$ be the sequence of eigenvalues of the operator A, repeated according to multiplicity: $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \to \infty \, (j \to \infty)$. Let $\{\mu_j\}$ be the sequence of eigenvalues of operator $A+V$, counted in the same way. The inverse spectral problem, analogous to that of [1], [2]—[4], [7], [9], is to show — under additional assumptions if necessary — that if the sequence $\{\mu_j\}$ is close in some sense (which will be defined later) to the sequence $\{\lambda_j\}$, then V is the zero operator, i.e., $q(x) = 0$ for almost all $x \in \Omega$.

3. Auxiliary results. Let A_t ($t \in \mathbf{R}$) denote the operator $A_t = A + tI$, where A is a given realization in $L^2(\Omega)$ of elliptic operator $\mathfrak{a}$ (see Section 2) and I is the identity operator on $L^2(\Omega)$. As we know, for sufficiently large $t \in \mathbf{R}$ the operator A_t is invertible, its inverse $A_t^{-1}\colon L^2(\Omega) \to L^2(\Omega)$ is a bounded operator which satisfies the estimate

$$(2) \qquad \|A_t^{-1}\| \leq Ct^{-1}, \quad \forall t \geq t_0,$$

where $C > 0$ is a constant independent of t. In the sequel we shall use the following regularity properties of the operator A.

Definition 1. The operator A, being a realization in $L^2(\Omega)$ of elliptic operator $\mathfrak{a}$, is said to have *the regularity property*: *of type I*, iff there exists an integer $k_0 > 1 + \frac{n}{m}$ such that $D(A^{k_0}) \subset H^{mk_0}(\Omega)$;

of type II, iff $m > \frac{n}{2}$ and $D(A) \subset H^m(\Omega)$.

Such properties were examined in author's paper [8].

The following Lemma is another version of its main results.

LEMMA 1. *If the operator A has the regularity property of type I* [*resp. of type II*], *then for sufficiently large $t \in \mathbf{R}$ the Green operator $G_t^k = (A+tI)^{-k}$, where $k = k_0$* [*resp. $k \geq 2$*], *is an integral operator*:

$$G_t^k u = \int_\Omega g_t^{(k)}(\cdot, y)u(y)\,dy, \quad \forall u \in L^2(\Omega),$$

with a continuous and bounded kernel $g_t^{(k)}\colon \Omega\times\Omega\to \boldsymbol{R}$ *having the asymptotic property*

$$\text{(3)}\qquad \lim_{t\to\infty} t^{k-\frac{n}{m}} g_t^{(k)}(x,y) = \delta_{x,y} w_k(x), \quad \forall x, y\in\Omega,$$

where

$$\text{(4)}\qquad w_k(x) = (2\pi)^{-n}\int_{\boldsymbol{R}^n} [(-1)^{m_0}\mathfrak{a}_0(x,\xi)+1]^{-k} d\xi,$$

and

$$\delta_{x,y} = \begin{cases} 1, & \text{if } x=y\\ 0, & \text{if } x\neq y.\end{cases}$$

Moreover, there is a constant $M>0$ *such that*

$$\text{(5)}\qquad |t^{k-\frac{n}{m}} g_t^{(k)}(x,y)| < M, \quad \forall x, y\in\Omega,\ \forall t\geqslant t_0.$$

LEMMA 2. *Suppose that the operator* A *(being a realization of differential operator* $\mathfrak{a}$*) has the regularity property of type I or of type II. Then there exists a constant* $K>0$, *depending on coefficients of principal part* $\mathfrak{a}_0$ *of operator* $\mathfrak{a}$, *such that*

$$\lambda_j = Kj^{\frac{m}{n}} + o(j^{\frac{m}{n}}) \quad \text{when } j\to\infty.$$

The proof of Lemma 2 is quite similar to that of [2] and therefore is omitted.

We shall make use of the following important facts as well.

THEOREM 1 (Bochenek [4]). *Let* H *be a real separable Hilbert space. Let* T, V *be self-adjoint linear operators from* H *into* H. *Suppose that the operator* T *is positive,* V *is bounded and* T^{-1}, $(T+V)^{-1}$ *are compact. Let* $\{\lambda_j\}$, $\{\mu_j\}$ *denote the sequences of eigenvalues (repeated according to the multiplicities) of operators* T *and* $T+V$, *respectively. If there exist constants* $K, \varkappa>0$ *such that*

$$\text{(6)}\qquad \lambda_j = Kj^{\varkappa}+o(j^{\varkappa}), \quad \text{as } j\to\infty,$$

and if the series

$$\sum_{j=1}^{\infty} j^{-1}|\mu_j-\lambda_j|$$

is convergent, then for each integer $k>1+\varkappa^{-1}$ *the operator* $VT_\varrho^{-k} = V(T+\varrho I)^{-k}$, *where* $\varrho\in\boldsymbol{R}^+$, *is a nuclear operator and for every* $t_0\geqslant 0$ *we have*

$$\text{(7)}\qquad \lim_{\varrho\to\infty} (t_0+\varrho)^{k-\varkappa^{-1}}\, tr(VT_\varrho^{-k}) = 0.$$

Here $tr(VT_\varrho^{-k})$ stands for the trace of operator $B = VT_\varrho^{-k}$, that means,

$$tr(B) = \sum_{j=1}^{\infty} \langle B\varphi_j, \varphi_j\rangle,$$

where $\{\varphi_j\}$ denotes any orthonormal basis in H.

THEOREM 2 (Kato [6], Th.IV.1.16). *Let T, V be linear operators acting in the Hilbert space H. Suppose that T is bounded [compact] and V is T-bounded, i.e., $D(T) \subset D(V)$ and there are constants $\beta, \gamma > 0$ such that*

$$(8) \qquad ||Vu|| \leqslant \beta||u|| + \gamma||Tu||, \qquad \forall u \in D(T).$$

If the constants β, γ satisfy the condition

$$(9) \qquad \beta||T^{-1}|| + \gamma < 1,$$

then $T+V$ is invertible and $(T+V)^{-1}$ is bounded [compact].

LEMMA 3 (Gould [5], Ch.IV.11). *Assume that T is a self-adjoint linear operator acting in Hilbert space H. If $u_0 \in D(T)$ is a normed vector such that*

$$\inf\{\langle Tu, u\rangle : u \in D(T), ||u|| = 1\} = \langle Tu_0, u_0\rangle = \lambda_0 \in \boldsymbol{R},$$

then u_0 is an eigenvector of operator T corresponding to the eigenvalue λ_0.

4. Introductory theorem. Now we shall prove a basic theorem that is an initial point to discuss the inverse spectral problem.

THEOREM 3. *Let A be a realization in $L^2(\Omega)$ of elliptic operator $\mathfrak{a}$ having the regularity property of type I [of type II] (see Def. 1) and let V be a multiplier operator q, with $q \in L^\infty(\Omega)$. Let $\{\lambda_j\}$, $\{\mu_j\}$ denote the sequences of eigenvalues of operators A and $A+V$, respectively (counted according to their multiplicities). If the series*

$$(10) \qquad \sum_{j=1}^{\infty} j^{-1}|\mu_j - \lambda_j|$$

is convergent, then for $k = k_0$ [for $k \geqslant 2$] *the equality*

$$(11) \qquad \int_\Omega w_k(x) q(x)\, dx = 0$$

holds, where the function $w_k \colon \Omega \to \boldsymbol{R}$ is defined by formula (4).

Proof. Note first that we can choose a number $t_0 \in \boldsymbol{R}$ so large that the operators $T := A_{t_0}$ and V will satisfy all assumptions of Theorem 1. Indeed, since A is bounded from below, it follows that for sufficiently large t the operator A_t is positive. Furthermore, A_t^{-1} is a compact operator. By Theorem 2, among these t a number t_0 can be chosen such that also $(T+V)^{-1}$ is compact. Namely, putting $\beta = ||V||$, $0 < \gamma < 1$ we see that inequality (8) is valid, and inequality (9) follows from the estimate (2). Besides, on the strength of Lemma 2 we have for $\lambda'_j = \lambda_j + t_0$ the asymptotic property (6) with $\varkappa = \dfrac{m}{n}$. The convergence of series (10) implies the same property for $\lambda'_j = \lambda_j + t_0$, $\mu'_j = \mu_j + t_0$. Consequently all assumptions of Theorem 1 are satisfied. Let $\{\varphi_j\}$ be the orthonormal sequence of eigenfunctions of operator A corresponding to eigenvalues λ_j. It is known that $\{\varphi_j\}$ form an orthonormal basis in $L^2(\Omega)$. The functions φ_j are also eigenfunctions of operator

$T_\varrho = T+\varrho I (\varrho \geqslant 0)$ corresponding to eigenvalues $\lambda_j+t_0+\varrho = \lambda'_j+\varrho$. Expressing the trace of operator VT_ϱ^{-k} by the basis $\{\varphi_j\}$ and putting $t = t_0+\varrho$, we obtain

$$(12)\qquad tr(VT_\varrho^{-k}) = \sum_{j=1}^{\infty} \langle VT_\varrho^{-k}\varphi_j, \varphi_j\rangle = \sum_{j=1}^{\infty} (\lambda'_j+\varrho)^{-k}\langle V\varphi_j, \varphi_j\rangle = \sum_{j=1}^{\infty} \int_\Omega (\lambda_j+t)^{-k}\varphi_j^2(x)q(x)\,dx\,.$$

On the other hand, in view of Lemma 1 we obtain by the well-known Mercer expansion theorem that

$$(13)\qquad \sum_{j=1}^{\infty} (\lambda_j+t)^{-k}\varphi_j^2(x)q(x) = g_t^{(k)}(x, x)q(x)\,, \qquad \forall x \in \Omega\,,$$

where $g_t^{(k)}\colon \Omega\times\Omega \to \boldsymbol{R}$ is the integral kernel of the operator $T_\varrho^{-k} = A_t^{-k}$. Moreover, for each positive integer N we have

$$\left|\sum_{j=1}^{N} (\lambda_j+t)^{-k}\varphi_j^2(x)q(x)\right| \leqslant |q(x)| \sum_{j=1}^{N} (\lambda_j+t)^{-k}\varphi_j^2(x) \leqslant \leqslant |q(x)|\, g_t^{(k)}(x, x) \leqslant M', \qquad \forall x \in \Omega\,,$$

since $\lambda_j+t>0$ $(j\in \boldsymbol{N})$ and the function q is essentially bounded. Hence, by Lebesgue's dominated convergence theorem applied to (13) we obtain that

$$(14)\qquad \sum_{j=1}^{\infty} \int_\Omega (\lambda_j+t)^{-k}\varphi_j^2(x)q(x)\,dx = \int_\Omega g_t^{(k)}(x, x)q(x)\,dx\,.$$

Next, the asymptotic properties (as $t\to\infty$) of function $g_t^{(k)}$ described by (3), (5) allow us to apply Lebesgue's theorem, so we have

$$(15)\qquad \lim_{t\to\infty} t^{k-\frac{n}{m}} \int_\Omega g_t^{(k)}(x, x)q(x)\,dx = \int_\Omega w_k(x)q(x)\,dx\,,$$

where the function w_k is defined by (4). Combining formulas (12), (14), (15) we obtain

$$(16)\qquad \lim_{\varrho\to\infty} (t_0+\varrho)^{k-\frac{n}{m}} tr(VT_\varrho^{-k}) = \int_\Omega w_k(x)q(x)\,dx\,.$$

The equality (11) follows now from (7) and (16).
This completes the proof.

5. Analysis of inverse spectral problem. We will now proceed to establish some additional conditions which ensure that $V = 0$, that is, $q = 0$ almost everywhere in the set Ω. As a first result we have:

THEOREM 4. *In addition to the hypotheses of Theorem* 3 *suppose also that the function* q *is nonnegative* [*nonpositive*] *on the set* Ω. *Then* $q = 0$ *on* Ω.

Proof. Since in view of assumption (1) the function w_k is positive on Ω, it follows that the function $w_k q$ is nonnegative [nonpositive] on Ω. Therefore by the formula (11) we have $w_k q = 0$ on Ω, thus $q = 0$ on Ω and the proof is complete.

In the following theorem we make weaker assumptions on the function q at the cost of strengthening the assumptions on elliptic operator $\mathfrak{a}$.

THEOREM 5. *In addition to the hypotheses of Theorem 3 assume also that*

1° *the principal part $\mathfrak{a}_0$ of operator $\mathfrak{a}$ is such that $w_k(x) \equiv W_k =$ const,*

2° *there exists a self-adjoint linear operator B acting in $L^2(\Omega)$ such that*

a) *all constant functions belong to the nullspace $N(B)$ of the operator B,*

b) *the operator $B+V$ is nonnegative.*

Then $q = 0$ on Ω.

Proof. By assumption 1° it follows from (11) that

$$\int_{\Omega} q(x)\,dx = 0\,.$$

Thus the normed constant function $u_0 := |\Omega|^{-\frac{1}{2}}$, where $|\Omega|$ denotes the measure of the set Ω, satisfies by assumption 2° a) the equation

$$\langle (B+V)u_0, u_0\rangle = \langle qu_0, u_0\rangle = \int_{\Omega} q(x)|\Omega|^{-1}dx = 0\,. \tag{17}$$

On the other hand, the assumption 2° b) implies the inequality

$$\langle (B+V)u, u\rangle \geqslant 0\,, \quad \forall u \in D(B) = D(B+V)\,. \tag{18}$$

It follows from (17) and (18) that

$$\inf\{\langle (B+V)u, u\rangle : \|u\| = 1\} = \langle (B+V)u_0, u_0\rangle = 0\,.$$

Using Lemma 3 we see that the number 0 is the first eigenvalue of operator $B+V$ corresponding to eigenfunction $u_0 = |\Omega|^{-\frac{1}{2}}$. This means that $(B+V)u_0 = 0$, but $(B+V)u_0 = q|\Omega|^{-\frac{1}{2}}$, so that $q = 0$. Therefore the proof is complete.

In the next theorem the additional assumptions on function q are replaced by the assumption on the first eigenvalue of operator $A+V$.

THEOREM 6. *In addition to the hypotheses of Theorem 3 suppose also that*

1° $w_k(x) \equiv W_k =$ *const,*

2° $\alpha_0(x) \equiv 0$,

3° *all constant functions belong to $D(A)$,*

4° $\mu_1 \geqslant 0$.

Then $q = 0$ on Ω.

Proof. It follows from assumptions 2°, 3° that all constant functions belong to $N(A)$. Next, the assumption 4° implies that the operator $A+V$ is nonnegative.
Thus we can apply Theorem 5 taking $B = A$ to obtain the desired result.

Remark 1. The functions w_k are constant in particular if the principal part $\mathfrak{a}_0$ of operator $\mathfrak{a}$ has constant coefficients.

Remark 2. In the considered inverse spectral problem in general it is impossible not to make any additional assumptions on the function q. To see this, take

$$\Omega = \{x = (x_1, \dots, x_n): |x| < 1\} \text{ and } \mathfrak{a} := (-\Delta)^{m_0} + \alpha_0 ,$$

where Δ is the Laplace operator and $\alpha_0(x) := x_1^3$. Let $q(x) := -2x_1^3$; evidently $q \neq 0$ (but the function q changes the sign). Let A be a realization in $L^2(\Omega)$ of operator $\mathfrak{a}$ associated with Dirichlet problem. That is, we define $D(A)$ to be the closure in $H^m(\Omega)$ of the set

$$\left\{u \in C^\infty(\bar{\Omega}): \frac{\partial^j u}{\partial v^j} = 0 \quad \text{on } \partial\Omega \ (j = 1, \dots, m_0)\right\},$$

where v stands for the unit inner normal vector to $\partial\Omega$. Note that the operators A and $A+V$ have the same eigenvalues λ_j corresponding to eigenfunctions φ_j and ψ_j respectively, where $\varphi_j(x_1, x_2, \dots, x_n) = \psi_j(-x_1, x_2, \dots, x_n)$. In this case all assumptions of Theorem 3 are satisfied and in accordance to its thesis we have $\int_\Omega q(x)dx = 0$, while $q \neq 0$.

Remark 3. The nearness condition for the sequences $\{\lambda_j\}$ and $\{\mu_j\}$ defined by the convergence of the series

$$\sum_{j=1}^{\infty} j^{-1}|\mu_j - \lambda_j| \tag{19}$$

is optimal in the sense that among the conditions $C(\sigma)$ consisting in the convergence of the series

$$\sum_{j=1}^{\infty} j^{-\sigma}|\mu_j - \lambda_j| , \tag{20}$$

with different $\sigma \in \boldsymbol{R}$, the condition $C(1)$ is a most general one ensuring the validity of Theorem 3 and hence of Theorems 4—6 as well. Indeed, the convergence of series (20) with any $\sigma < 1$ implies the convergence of series (19). On the other hand, the condition $C(\sigma)$ with any $\sigma > 1$ does not suffice for the equality (11). To see this, take $q(x) \equiv Q > 0$. Then $\mu_j - \lambda_j = Q$, so the series (20) is convergent with any $\sigma > 1$, but $\int_\Omega w_k(x)q(x)dx > 0$.

6. Bibliographic comment. The proofs of Theorems 3—6 are based on the methods of Bochenek's papers [2]—[4]. In these works one of the main steps is developping an asymptotic formula concerning eigenvalues and eigenfunctions of given boundary value problem. In our situation a counterpart to this formula can be expressed as follows: if the function $\Phi_k(x, t)$ is defined by

$$\sum_{j=1}^{\infty} (\lambda_j + t)^{-k}\varphi_j^2(x) = w_k(x)t^{\frac{n}{m}-k} - \Phi_k(x, t) , \tag{21}$$

then for $k > \frac{n}{m}$ we have

$$(22) \qquad \int_{\Omega} |\Phi_k(x, t)|\, dx = o(t^{\frac{n}{m}-k}), \quad \text{as } t \to \infty .$$

We are in a position to obtain easily the formula (22).

In fact, combining the equation (13) with $q = 1$ and the formula (21) we see that

$$t^{k-\frac{n}{m}} \Phi_k(x, t) = w_k(x) - t^{k-\frac{n}{m}} g_t^{(k)}(x, x), \quad \forall x \in \Omega .$$

Next, it follows from (3) and (5) that

$$\lim_{t \to \infty} t^{k-\frac{n}{m}} \int_{\Omega} |\Phi_k(x, t)|\, dx = 0 ,$$

so (22) is true. In view of this fact, the suitable part of proof of Theorem 3 can be copied from [2]—[4], however the presented version seems to be relatively simpler and clearer.

References

[1] V. A. Ambarzumian, *Über eine Frage der Eigenverttheorie*, Zeitschr. Phys., 53 (1929), 690—695.

[2] J. Bochenek, *Zależność równania różniczkowego cząstkowego od wartości własnych odpowiedniego zagadnienia*, Zeszyty Naukowe PK, 15 (1968), 3—54.

[3] J. Bochenek, *On the inverse problem of the Sturm-Liouville type for a linear partial differential equation with constant coefficients of the second order*, Ann. Polon. Math., 24 (1971), 331—341.

[4] J. Bochenek, *Global inverse problem of the Sturm-Liouville type for a linear elliptic partial differential equation of second order*, Ann. Polon. Math., 40 (1983), 271—281.

[5] S. H. Gould, *Variational Methods for Eigenvalue Problems*, London 1966.

[6] T. Kato, *Perturbation theory for Linear Operators*, Berlin 1966.

[7] Н. В. Кузнецов (N. V. Kuznecov). *Обобщение одной теоремы В. А. Амбарцумяна*, ДАНСССР 146 (1962), 1259—1262.

[8] W. Pielichowski, *Asymptotic behaviour of iterated Green's functions of a linear elliptic differential operator*, this fascicle.

[9] S. Postawa, *On the inverse problem of the Sturm-Liouville type for some class of the partial differential equations of the fourth order*, Universitatis Iagellonicae Acta Mathematica (in press).

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