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On the existence of polynomial connection valued concomitants

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§ 1. Introduction.

Let $\mathscr{K} \in \otimes\, \boldsymbol{R}^{n} \otimes \otimes\, (\boldsymbol{R}^n)^*$ and let G be a transformation group of $\boldsymbol{R}^n$ which preserves $\mathscr{K}$ i.e. the isotropy group at $\mathscr{K}$. For any differentiable manifold M of dimension n and for any G-structure P on M there exists a tensor field $K \in \mathscr{T}^r_s M$ defined by $\mathscr{K}$. Namely, the coefficients of $u{*}K$ at x, where $u \in P_x$, are equal to the coefficients of $\mathscr{K}$ and they do not depend on u. The pair (M, K) is called a structure induced by the tensor $\mathscr{K}$.

The object of interest in this note are linear connection valued concomitants of certain structures (M, K) as above. The following definitions will be useful

$\mathscr{S}\mathscr{K}_n$:= the set of all pairs (M, K), where $\dim M = n$ and K is a tensor field on M induced by $\mathscr{K}$;

$\mathscr{L}\mathscr{C}_n$:= the set of all pairs (M, ∇), where $\dim M = n$ and ∇ is a linear connection on M.

Definition 1. A map $\mathscr{C}$ from the set $\mathscr{S}\mathscr{K}_n$ into the set $\mathscr{L}\mathscr{C}_n$ is called a connection valued concomitant of structure induced by the tensor $\mathscr{K}$ if

(i) for every $(M, K) \in \mathscr{S}\mathscr{K}_n$ the underlying manifold of the pair $\mathscr{C}(M, K)$ is M (hence we denote $\mathscr{C}(M, K) = (M, C(K))$);

(ii) there exist functions $C^i_{jk}\colon \boldsymbol{R}^N \to \boldsymbol{R}$ of class C^∞ $(i, j, k = l, \ldots, n)$ of variables $v^{a_1 \ldots a_r}_{b_1 \ldots b_s}$; $v^{a_1 \ldots a_r}_{b_1 \ldots b_s c_1}$; $\ldots$; $v^{a_1 \ldots a_r}_{b_1 \ldots b_s c_1 \ldots c_q}$ $(a, b, c = l, \ldots, n)$, which are such that if $(M, K) \in \mathscr{S}\mathscr{K}_n$ and $x = (x^i)$ is a chart of M, then the x-components of $C(K)$ are given by

$$[C(K)]^i_{jk} = C^i_{jk}(K^{a_1 \ldots a_r}_{b_1 \ldots b_s};\ K^{a_1 \ldots a_r}_{b_1 \ldots b_s, c_1};\ \ldots;\ K^{a_1 \ldots a_r}_{b_1 \ldots b_s c_1 \ldots c_q}).$$

If $\mathscr{C}$ is a connection valued concomitant and it is possible to choose C^i_{jk} in (ii) so that the highest occurring derivative of $K^{a_1 \ldots a_r}_{b_1 \ldots b_s}$ appearing in them is of q-th order, then $\mathscr{C}$ is said to be of q-th order. If the functions C^i_{jk} are polynomial functions, then the concomitant $\mathscr{C}$ is called polynomial.

We shall be concerned with the problem of the existence of polynomial connection valued concomitants induced by any tensor of type (1,1) or (0,2). Such a concomitant exists e.g. in the case of pseudoriemannian structure. On the other hand, A. Zajtz proved in [5] a theorem which states that the existence of the connection valued concomitant of the first order implies $g^{(1)} = 0$, where $g = \text{Lie } G$ and $g^{(1)}$ denotes the first prolongation of g. The authors of [1] proved that the connection valued concomitants (not necessarily polynomial) of almost complex, almost product and almost tangent structures do not exist. In this note we want to give a certain generalization of the results from [5] and [1]. Applying the method from [1], we prove that the theorem of A. Zajtz is still true in the case of arbitrary order, if the concomitant is polynomial and induced by (1, 1) or (0, 2)-tensor.

Remark. D. B. A. Epstein in [2] defined a natural connection of Riemannian manifolds. Following that definition, a map $\mathscr{C}$ from $\mathscr{S}\mathscr{K}_n$ to $\mathscr{L}\mathscr{C}_n$ is called a natural connection if

(i) for every $(M, K) \in \mathscr{S}\mathscr{K}_n$ the underlying manifold of $\mathscr{C}(M, K)$ is M, i.e. $\mathscr{C}(M, K) = (M, C(K))$;

(ii) if U is open then $C(M, K)|U = C(U, K|U)$;

(iii) if $(M, K), (N, L) \in \mathscr{S}\mathscr{K}_n$ and $f\colon M \to N$ is a diffeomorphism such that $f_* K = L$ then $f_* C(M, K) = C(N, L)$.

One can see that a connection valued concomitant is a natural connection, but it is not known whether the converse is true.

§ 2. Local form of polynomial concomitants

In this section we shall describe a local form of polynomial connection valued concomitants of structures induced by any (1, 1)-tensor or (0, 2)-tensor. First we prove the following

Proposition 1. (a) *If* $(M, A) \in \mathscr{S}\mathscr{A}_n$, *where* $\mathscr{A} \in \mathbf{R}^n \otimes (\mathbf{R}^n)^*$, *then every polynomial connection valued concomitant has a local form which is a linear combination of terms of the form* $A^a_{b,c} A^{a_1}_{b_1} \dots A^{a_m}_{b_m}$. (b) *There does not exist a polynomial connection valued concomitant of a structure induced by a* (0, 2)*-tensor.*

Proof. Let $\mathscr{C}$ be such a concomitant and let $x = (x^i)$ and $\bar x = (\bar x^i)$ be two charts of M. Definition 1 implies that the x and $\bar x$ components of $C(A)$ must be related by

$$C^i_{jk}(\bar A^a_b; \bar A^a_{b,c_1}; \dots \bar A^a_{b,c_1\dots c_q}) = [\overline{C(A)}]^i_{jk} =$$
$$= C^\alpha_{\beta\gamma}(A^a_b; A^a_{b,c_1}; \dots; A^a_{b,c_1\dots c_q}) \frac{\partial \bar x^i}{\partial x^\alpha} \frac{\partial x^\beta}{\partial \bar x^j} \frac{\partial x^\gamma}{\partial \bar x^k} + \frac{\partial^2 x^\alpha}{\partial \bar x^j \partial \bar x^k} \frac{\partial \bar x^i}{\partial x^\alpha}. \tag{2.1}$$

We fix an arbitrary chart x and for any $t > 0$ we define a new chart $\bar x$ by $x^i = t\bar x^i$. Hence equation (2.1) reduces to

$$t[C(A)]^i_{jk} = C^i_{jk}(A^a_b;\ tA^a_{b,c_1}; \dots; t^q A^a_{b,c_1\dots c_2}). \tag{2.2}$$

Assume that functions C^i_{jk} in (2.2) are polynomial. After differentiating (2.2) once with respect to t and taking the limit as $t \to 0+$, one can see that C^i_{jk} must be a linear combination of terms of the form $A^a_{b,c_1} A^{a_1}_{b_1} \dots A^{a_m}_{b_m}$.

(b) Suppose that such a concomitant exists. Using an argument similar to the one used to prove (a) we obtain that in any chart $C^i_{jk} \equiv 0$. But it is a contradiction with the transformation rule of linear connection.

However, in the case where the $(0, 2)$-tensor $\mathscr{F}$ which induces the structure (M, F) is non-degenerate we may consider a more natural concept of concomitants. We shall assume that local functions from Definition 1 are of variables v_{ab}; v^{ab}; v_{ab,c_1}; ...; $v_{ab,c_1 \dots c_q}$ and we shall put coordinates F^{ab} in place of additional variables v^{ab} (F^{ab} are such that $F^{ab}F_{bc} = F_{cb}F^{ba} = \delta^a_c$). From Terng's paper [4] we know that such concomitants are equivariant to rational concomitants of F.

It will be necessary to put forward the following

Definition 2. A polynomial connection valued concomitant $\mathscr{C}$ of the structure induced by $\mathscr{F}$ is called homogeneous if there exists $w \in \boldsymbol{R}$ such that $C(\lambda^2 F) = \lambda^W C(F)$ for every $\lambda > 0$. Then w is called a weight of $\mathscr{C}$.

LEMMA *If* $\mathscr{C}$ *polynomial connection valued concomitant of structure induced by non-degenerate* $(0, 2)$*-tensor* $\mathscr{F}$ *is homogeneous, then the weight of* $\mathscr{C}$ *is* 0.

Proof. Since the concomitant $\mathscr{C}$ is polynomial and due to Definition 2 the weight of $\mathscr{C}$ is an even integer $2w$. Let x be the identical chart of $\boldsymbol{R}^n$. We shall consider a constant structure $(\boldsymbol{R}^n, F)$, i.e. a structure F on $\boldsymbol{R}^n$ whose x-components at any point are equal to those of $\mathscr{F}$. From Definition 1 we have

$$[C(F)]^i_{jk} = C^i_{jk}(F_{ab});\ F^{ab};\ 0;\ \dots;\ 0)\,. \tag{2.3}$$

Then (2.3) and the homogeneity of $\mathscr{C}$ yield

$$\lambda^{2w}[C(F)]^i_{jk} = C^i_{jk}(\lambda^2 F_{ab};\ \lambda^{-2} F^{ab};\ 0;\ \dots;\ 0)\,. \tag{2.4}$$

Defining a new chart y by $x^i = \lambda y^i$ and using (2.3) we obtain for any $\lambda > 0$

$$\lambda\,[C(F)]^i_{jk} = C^i_{jk}(\lambda^2 F_{ab};\ \lambda^{-2} F^{ab};\ 0;\ \dots;\ 0)\,. \tag{2.5}$$

Upon combining equations (2.4) and (2.5) we have $[C(F)]^i_{jk} \equiv 0$. Now we introduce another chart z by $z^i = x^i - \frac{1}{2}\Sigma x^j x^k$ on a neighborhood of $0 \in \boldsymbol{R}^n$. By the transformation rule

$$\overline{[C(F)]}^i_{jk} = C^\alpha_{\beta\gamma}(F_{ab};\ F^{ab};\ F_{ab,c_1};\ \dots;\ F_{ab,c_1 \dots c_q}) \frac{\partial z^i}{\partial x^\alpha} \frac{\partial x^\beta}{\partial z^j} \frac{\partial x^\gamma}{\partial z^k} + \frac{\partial^2 x^\alpha}{\partial z^j \partial z^k} \frac{\partial z^i}{\partial x^\alpha} \tag{2.6}$$

(the bar denotes z-component) we find that the z-components of $C(F)$ at 0 are equal 1. Since $[C(\overline{\lambda^2 F})]^i_{jk} = \overline{[C(\lambda^2 F)]}^i_{jk} = \lambda^{2w}\overline{[C(F)]}^i_{jk}$, we get at 0

$$[C(\overline{\lambda^2 F})]^i_{jk} = 1\,. \tag{2.7}$$

Then, since $\lambda^2\bar{F} = \overline{\lambda^2 F}$ and since $\mathscr{C}$ is homogeneous, we have

$$[C(\overline{\lambda^2 F})]^i_{jk} = [C(\lambda^2\bar{F})]^i_{jk} = \lambda^{2w}[C(\bar{F})]^i_{jk} = \lambda^{2w}[\overline{C(F)}]^i_{jk}$$

and hence at 0

$$[C(\overline{\lambda^2 F})]^i_{jk} = \lambda^{2w}. \tag{2.8}$$

Finally, from equations (2.8) and (2.7), $2w = 0$.

PROPOSITION 2. *In the case of a structure* (M, F) *induced by a non-degenerate* $(0, 2)$*-tensor* $\mathscr{F}$ *a homogeneous polynomial connection valued concomitant has a local form which is a linear combination of terms of the form* $F_{ab,c}F_{a_1b_1}\dots F_{a_{m-1}b_{m-1}}F^{a_mb_m}\dots F^{a_{2m-1}b_{2m-1}}$.

Proof. Let $\mathscr{C}$ be such a concomitant and let x be a chart of M. In the equation

$$[C(F)]^i_{jk} = C^i_{jk}(F_{ab};\, F^{ab};\, F_{ab,c_1};\, \dots;\, F_{ab,c_1\dots c_q})$$

we put $\lambda^{-2}F$ instead of F and we obtain

$$[C(F)]^i_{jk} = C^i_{jk}(\lambda^{-2}F_{ab};\, \lambda^2F^{ab};\, \lambda^{-2}F_{ab,c};\, \dots;\, \lambda^{-2}F_{ab,c_1\dots c_q}),$$

since the weight of $\mathscr{C}$ is 0. Observe that the above equation implies that in any monomial of C^i_{jk} a number of variables with upper indices is equal to a number of variables with lower indices. Then for any $\lambda > 0$ we define a new chart y by $x^i = \lambda y^i$ and we write the above equation in it

$$\lambda[C(F)]^i_{jk} = C^i_{jk}(F_{ab};\, F^{ab};\, \lambda F_{ab,c_1};\, \dots;\, \lambda^q F_{ab,c_1\dots c_q}).$$

We conclude the proof by an argument similar to that used in the proof of Proposition 1 (a).

Problem 1. Is it possible to omit the assumption of homogeneity in Lemma and in Proposition 2?

Problem 2. Epstein in [2] has shown that any concomitant (even of infinite order) of Riemannian structures is polynomial of finite order. Then, Aldersley, Horndeski and Mess [1] have recently proved that any tensorial valued or linear connection valued concomitant of almost complex structure is polynomial. The problem is whether it is true for concomitants of structures induced by any tensor of type (1,1) or (0,2).

§ 3. Existence of connection valued concomitants

In this section we shall give a negative answer on the existence of connection valued concomitants of a broad class of structures. Let us recall that if g is a Lie subalgebra of $gl(n)$ then the first prolongation of g is defined to be $g^{(1)} = g\otimes \mathbf{R}^{n*} \cap \mathbf{R}^n\otimes S^2(\mathbf{R}^{n*})$

THEOREM. *Let* $G \subset GL(n)$ *denote the isotropy group of* $\mathscr{A}$ (*respectively* $\mathscr{F}$) *and let us denote* $g = \mathrm{Lie}\, G$. *If there exists a polynomial* (*resp. homogeneous polynomial*) *connection valued concomitant of structures induced by* (1,1)*-tensor* $\mathscr{A}$ (*resp., by non-degenerate* (0,2)*-tensor* $\mathscr{F}$) *then* $g^{(1)} = 0$.

Proof. We shall prove only the case of structures induced by $\mathcal{F}$; the proof of another case as analogous will be omitted.

Suppose $\mathcal{C}$ to be a connection valued concomitant of structures induced by $\mathcal{F}$. Proposition 2 tells us that the functions $[C(F)]^i_{jk}$ can be expressed as a linear combination of terms of the form $F_{ab,c}F_{a_1b_1} \dots F_{a_{m-1}b_{m-1}}F^{a_m b_m} \dots F^{a_{2m-1}b_{2m-1}}$ (a, b, c are fixed in each term and may be different in different terms; m also depends on term). We consider a constant structure $(\boldsymbol{R}^n, F)$ i.e. the coefficients of F are equal to those of $\mathcal{F}$ in the identity chart x of $\boldsymbol{R}^n$. Therefore from the above local form we see that all $[C(F)]^i_{jk}$ vanish.

Let $\mathcal{S}$ denote the vector subspace of $\boldsymbol{R}^{n^3}$ which consists of all collections of real numbers $\{S^i_{jk}\}$ such that $S^i_{jk} = S^i_{kj}$. For any $\{S^i_{jk}\} \in \mathcal{S}$ we define a new chart $\bar{x}$ by $\bar{x}^i = x^i - \frac{1}{2}S^i_{jk}x^j x^k$ on a neighborhood of $0 \in \boldsymbol{R}^n$. Using the transformation rule (2.6) we have at 0

$$\overline{[C(F)]}{}^i_{jk} = S^i_{jk} \tag{4.1}$$

since $[C(F)]^i_{jk}$ vanish. On the other hand, $\overline{[C(F)]}{}^i_{jk}$ is the same linear combination of the same terms as above only with bars over F. Thus we discover that S^i_{jk} is a linear combination of terms as above.

It will be necessary to apply the transformation law for F in the form

$$\bar{F}_{ab}\frac{\partial \bar{x}^a}{\partial x^i}\frac{\partial \bar{x}^b}{\partial x^j} = F_{ij}. \tag{4.2}$$

By substituting the coordinates $\bar{x}^i = x^i - \frac{1}{2}S^i_{jk}x^j x^k$ in (4.2) we have at 0

$$\bar{F}_{ab}(\delta^a_i - S^a_{is}x^s)(\delta^b_j - S^b_{jt}x^t) = F_{ij}. \tag{4.3}$$

Therefore at 0

$$F_{ab} = \bar{F}_{ab}. \tag{4.4}$$

Then we differentiate equation (4.3) with respect to x^k at 0 and we obtain

$$F_{ij,k} = \bar{F}_{ib}S^b_{jk} + \bar{F}_{aj}S^a_{ik}. \tag{4.5}$$

Finally, from (4.4) and (4.5) we obtain that each S^i_{jk} is a linear combination of terms of the form

$$(F_{ad}S^d_{bc} + F_{db}S^d_{ac})F_{a_1b_1} \dots F_{a_{m-1}b_{m-8}}F^{a_m b_m} \dots F^{a_{2m-1}b_{2m-1}} = T_{abc}(F_{ad}S^d_{bc} + F_{db}S^d_{ac}),$$

where a, b, c are fixed natural numbers such that $1 \leqslant a, b, c \leqslant n$, the components of F are taken at 0 and the coefficients T_{abc} depend obviously on i, j, k.

Now we are in position to conclude the proof. We define a linear map $f\colon \mathcal{S} \to gl(n) \oplus \dots \oplus gl(n)$ (n components), $f = (f_1, \dots, f_n)$, by

$$f_k(S) = S^t_k F + F S_k \quad (k = 1, \dots, n),$$

where $S_k \in gl(n)$ such that $(S_k)_{ij} = S^i_{jk}$. Next we define another linear map $h\colon gl(n) \oplus \dots \oplus gl(n) \to \mathcal{S}$ such that $h \circ f =$ identity on $\mathcal{S}$. Namely, from the above statement we see that for any i, j, k there are $T = (T_{abc})$ such that

$$S^i_{jk} = \sum_{a,b,c=1}^{n} T_{abc}(S^t_c F + F S_c)_{ab},$$

so we can define

$$h(A_1, \dots, A_n) = \sum_{a,b,c=1}^{n} T_{abc} A^a_{bc},$$

where $A_c = [A^a_{bc}] \in gl(n)$.

On the other hand, we see that $\operatorname{Ker} f = g^{(1)}$, because $A \in g$ if and only if $A^t F + FA = 0$. But f is a monomorphism, so $g^{(1)} = 0$.

COROLLARY 1. *There does not exist a polynomial connection valued concomitant of a structure induced by a tensor of type* (1,1).

Proof. For each structure of this kind $g^{(1)} \neq 0$ (Guillemin [3]).

For example, there do not exist such concomitants in the cases of almost complex, almost product, almost tangent structures as well as in the case of almost multifoliate structure.

COROLLARY 2. *There does not exist a homogeneous polynomial connection valued concomitant of almost symplectic structure.*

In fact, it is well known that $sp(n)^{(k)} \neq 0$ for $k = 0, 1, 2, \dots$

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