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Natural topologies on R^n

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We denote by N the topology on R^n given by the norm

$$||x|| = (\sum_{i=1}^{n} (x_i)^2)^{1/2}$$

where $x = (x_1, \ldots, x_n)$ is a point from R^n. If $U, V \in N$, then a diffeomorphism $\varphi\colon U \to V$ of class C^∞ is called a local diffeomorphism of R^n. Let θ be a topology on R^n such that if f is a local diffeomorphism of R^n, then f gives a homeomorphism between $(\mathrm{dom}(f), \theta|\mathrm{dom}(f))$ and $(\mathrm{im}(f), \theta|\mathrm{im}(f))$. We call the topology θ as above a natural topology on R^n.

The main theorem is the following.

THEOREM. *If θ is a natural topology on R^n and has a countable basis, then $\theta = N$ or $\theta = \{R^n, \emptyset\}$.*

From now on, we consider θ a natural topology on R^n, with a countable basis. For each x from R^n, we denote by τ_x translation by x. We prove some lemmas.

LEMMA 1. *If $\theta \supset N$, then $\theta = N$.*

Proof. θ is Hausdorff. By Proposition 1.1 in [2] with $ER^n = (R^n, \theta)$, $\pi = \mathrm{id}\colon ER^n \to R^n$ and $Ef = f$ for each local diffeomorphism f of R^n, we have that the mapping

$$E\tau\colon R^n \times ER^n \to ER^n, \ E\tau(x, v) = E\tau_x(v)$$

is continuous. Hence the map

$$(R^n, N) \to (R^n, \theta), \qquad x \to x = E\tau(x, 0)$$

is continuous. Therefore $N \supset \theta$. The lemma is proved.

LEMMA 2. $\theta \subset N$.

Proof. Let θ^* be the minimal topology such that $\theta \cup N \subset \theta^*$. By Lemma 1 with θ^* playing the role of θ, $\theta^* = N$. Therefore $\theta \subset N$. The lemma is proved.

Assume $\theta \neq \{R^n, \emptyset\}$. Let V_p $(p = 1, 2, \ldots)$ be a countable basis of θ such that $V_p \neq \emptyset$ for all p. Denote by int_N the interior with respect to N.

LEMMA 3. *There exists a natural number p such that* $\mathrm{int}_N(R^n \setminus V_p) \neq \emptyset$.

Proof. There exist a point y_0 from R^n and a natural number p_0 such that $y_0 \in V_{p_0}$. By Lemma 2, $V_p \in N$ for all p. Suppose that for each p,

$$\mathrm{int}_N(R^n \setminus V_p) = \emptyset .$$

Then by the Baire category theorem [1] there exists $z_0 \in R^n$ such that $z_0 \in V_p$ for all p. Define

$$\varphi\colon (R^n, \theta) \to (R^n, \theta), \qquad \varphi(x) = y_0 + z_0 - x .$$

Since $\varphi(y_0) = z_0$, φ is not homeomorphic. This is a contradiction and so the lemma is proved.

Proof of the theorem. Assume $\theta \neq \{R^n, \emptyset\}$. Let V_p $(p = 1, 2, \ldots)$ be a countable basis of θ such that $V_p \neq \emptyset$ for all p. Applying Lemma 3, choose p such that $\mathrm{int}_N(R^n \setminus V_p) \neq \emptyset$. Choose $a \in R^n$ and $r > 0$ such that

$$\{x \in R^n\colon \|x - a\| < r\} \subset R^n \setminus V_p .$$

Putting

$$W = \left(\frac{1}{r}\,\mathrm{id}\right) \circ \tau_{-a}(V_p),$$

we find that $W \in \theta$, $W \neq \emptyset$ and

$$\{x \in R^n\colon \|x\| < 1\} \subset R^n \setminus W .$$

(Then $W \in \theta | R^n \setminus \{0\}$.) Let I be the inversion such that $I|\{\|x\| = 1\} = \mathrm{id}$. We see that $I(W) \in \theta | R^n \setminus \{0\}$, $I(W) \neq \emptyset$ and

$$I(W) \subset \{x \in R^n\colon \|x\| \leqslant 1\} \setminus \{0\} .$$

Since $I(\mathrm{W}) \in \theta$ or $I(W) \cup \{0\} \in \theta$, there exists a set $U \in \theta$ such that $U \neq \emptyset$ and $U \subset \{x \in R^n\colon \|x\| \leqslant 1\}$.

Let $x_0 \in R^n$ and $\varepsilon > 0$. Choose a point $y \in U$. Then

$$x_0 \in (\tau_{x_0}) \circ \left(\frac{\varepsilon}{3}\,\mathrm{id}\right) \circ (\tau_{-y})(U) \subset \{x \in R^n\colon \|x - x_0\| < \varepsilon\}$$

and

$$(\tau_{x_0}) \circ \left(\frac{\varepsilon}{3}\,\mathrm{id}\right) \circ (\tau_{-y})(U) \in \theta .$$

It follows that $\theta \supset N$. Therefore (by Lemma 2) $\theta = N$. The theorem is proved.

Example. Let θ be the minimal topology on R such that for each $(p, q) \in Q^2$, we have

$$(-\infty, p) \cup (q, \infty) \in \theta .$$

It is easy to see that θ has a countable basis, $\theta \neq N$ and $\theta \neq \{R, \emptyset\}$. Moreover, if $f\colon R \to R$ is a global diffeomorphism of class C^∞, then f gives a homeomorphism between (R, θ) and (R, θ).

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References

[1] R. Engelking, *General topology*, Warsaw 1977.

[2] W. M. Mikulski, *Locally determined associated spaces*, J. London Math. Soc., (2) 32 (1985), 357—364.

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