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Note on discontinuous solutions of a functional equation

BOGDAN CHOCZEWSKI

In the paper by M. Kuczma and the author [1] the linear homogeneous functional equation

$$\varphi(f(x)) = g(x)\varphi(x) \tag{1}$$

has been dealt with in an interval $I := (0, A]$ or $(0, A)$, where $0 < A \leqslant +\infty$. The following results on the asymptotic behaviour of solutions of (1) at the origin have been obtained in [1] (all the asymptotic symbols in the present paper refer to $x \to 0+$).

PROPOSITION. *Assume that*

(H_0) $f, g\colon I^* \to \mathbf{R}$ *are continuous on* $I^* := I \cup \{0\}$ *and* $0 < f(x) < x$, $g(x) > 0$ *in* I.

(H_1) $f(x) = x - ax^{m+1} + O(x^{m+1+\mu})$; $a, m, \mu > 0$.

(H_2) $g(x) = 1 + bx^k + O(x^{k+\varkappa})$; $b, k, \varkappa > 0$.

1° *If* $k = m$, *then every solution* $\varphi\colon I \to \mathbf{R}$ *of equation* (1) *continuous on* I *has the asymptotic property*

$$\varphi(x) = O(x^{-b/a}). \tag{2}$$

2° *If* $k < m$ *and* $\min(k, \mu, \varkappa) > m - k$, *then every solution* $\varphi\colon I \to \mathbf{R}$ *of* (1) *continuous on* I *has the property*

$$\varphi(x) = O(\exp([b/a(m-k)]x^{k-m})). \tag{3}$$

In the cases covered by the assumptions of the Proposition equation (1) has in I the continuous solution depending on an arbitrary function (cf. [2], Ch. II) but only the solution $\varphi(x) = 0$ for $x \in I$ may be extended to a solution continuous on I^* (cf. the Lemma in [1]).

Following the lines of concluding remarks in [1], in the present note we assume more about the asymptotic behaviour of the function g and obtain a result slightly improving on that quoted above.

Our assumptions on g read:

(H$_3$) *There are: a positive integer n, positive numbers k_i, b_i ($i = 1, \ldots, n$) and p such that $0 < k_1 < \ldots < k_n \leqslant m < k_n + p \leqslant 2k_1$ and*

$$g(x) = 1 + b_1 x^{k_1} + \ldots + b_n x^{k_n} + O(x^{k_n + p}). \tag{4}$$

We are going to prove the following theorem based on the subsequent lemma.

THEOREM. *Under hypotheses* (H$_0$), (H$_1$) (*with $\mu > m - k_1$ if $k_n < m$*) *and* (H$_3$) *every solution φ: $I \to \mathbf{R}$ continuous on I has the asymptotic property*

(i) *if $k_n = m$:*

$$\varphi(x) = O(x^{-b_n/a} \exp(\sum_{i=1}^{n=1} c_i x^{k_i - m})), \tag{5}$$

(ii) *if $k_n < m$:*

$$\varphi(x) = O(\exp(\sum_{i=1}^{n} c_i x^{k_i - m})), \tag{6}$$

where

$$c_i := b_i / a(m - k_i), \quad i = 1, \ldots, n. \tag{7}$$

LEMMA. *If a function g satisfies hypotheses* (H$_0$) *and* (H$_3$), *then there exist positive numbers $p_1, \ldots, p_{n-1}$ such that*

$$m < k_i + p_i \leqslant 2k_1 \tag{8}$$

and functions g_i fulfilling (H$_0$) *and having the properties*:

$$g(x) = g_1(x) \ldots g_n(x), \quad x \in I^*, \tag{9}$$

$$g_i(x) = 1 + b_i x^{k_i} + O(x^{k_i + p_i}); \ p_n := p, \quad i = 1, \ldots, n. \tag{10}$$

Proof of the Lemma. For $n = 1$ the Lemma follows from (H$_3$). Assume it to be true for functions $\tilde{g}$ satisfying (H$_0$) and having the property

$$\tilde{g}(x) = 1 + b_1 x^{k_1} + \ldots + b_{n-1} x^{k_{n-1}} + O(x^{k_{n-1} + r}) \tag{11}$$

with some $r > 0$ such that $m < k_{n-1} + r \leqslant 2k_1$.

Given a function g fulfilling (H$_0$) and (H$_3$) we take $r := p + k_n - k_{n-1}$, a function $\tilde{g}$ with properties (H$_0$) and (11) (with this r) and a function g_n having properties (H$_0$) and (10) for $i = n$ with the remainder to be appropriately determined. We then have

$$\tilde{g}(x) g_n(x) = 1 + b_1 x^{k_1} + \ldots + b_n x^{k_n} + [O(x^{k_{n-1} + r}) + \\ + O(x^{k_n + p}) + \sum_{j=1}^{n-1} b_j b_n x^{k_j + k_n} + O(x^{k_{n-1} + r + \varrho})] \tag{12}$$

with a positive ϱ. We see that the dominating term in square brackets above is $O(x^{k_n + p})$, as $k_{n-1} + r = k_n + p$ and $k_j + k_n > 2k_1 > k_n + p$. Thus we can made the remainder in (12) equal to that in (4) by a suitable choice of g_n fulfilling (10) for $i = n$, i.e. we can obtain

$g(x) = \tilde{g}(x) g_n(x)$. According to the inductive hypothesis we get formula (9) with the factors satisfying (10) and (8), as needed.

Proof of the Theorem. On account of the Lemma we can write g as the product of the factors g_i which satisfy (H_0) and (10) with (8). Consider the equations

$$(13_i) \qquad \varphi(f(x)) = g_i(x)\varphi(x), \quad i = 1, \dots, n.$$

We claim that the functions f and g_i have all the properties required for the validity of the Proposition for solution of equation (13_i). For, note that (10) correspond to (H_2) and (8) and (H_3) yield (if $k_n < m$): $p_i > m - k_i$; $2k_i > 2k_1 > m$; $\mu > m - k_1 > m - k_i$. Consequently $\min(k_i, \mu, p_i) > m - k_i$. From the Proposition we know that every solution $\varphi_i\colon I \to \mathbb{R}$ of (13_i) continuous on I has the property

$$(14) \qquad \varphi_i(x) = O(\exp(c_i x^{k_i - m})) \quad \text{for } i = 1, \dots, n-1,$$

where c_i are defined by (7), and solutions $\varphi_n\colon I \to \mathbb{R}$ of (13_n) continuous in I satisfy (14) with $i = n$ if $k_n < m$ whereas

$$(15) \qquad \varphi_n(x) = O(x^{-b_n/a})$$

if $k_n = m$.

Take some, positive and continuous in I, solutions φ_i of each equation (13_i). By (9) and (13_i) $(i = 1, \dots, n)$ the function $\varphi(x) := \varphi_1(x) \dots \varphi_n(x)$, $x \in I$, is a solution of (1), continuous and positive in I, possessing the properties (5) or (6) (according to whether (i) or (ii) of the Theorem holds), which follow from (14) and (15). (Note that $h_1 \in O(u(x))$ and $h_2 \in O(v(x))$ imply $h_1 h_2 \in O(u(x)v(x))$.) But all the solutions of (1) continuous on I have the same behaviour at the origin as φ. This results from the Theorem 1 in [1] and concludes the proof.

Remark. The Theorem, when applied e.g. to the equation

$$\varphi(x - x^3) = (1 + x^{3/2} + x^2 + O(x^3))\varphi(x), \; x \in [0, 1)$$

yields, for each of its solutions φ continuous on $(0, 1)$, the relation

$$\varphi(x) = O(x^{-1}\exp(2x^{-1/2})),$$

which cannot be obtained from the Proposition.

References

[1] B. Choczewski, M. Kuczma, *Asymptotic properties of discontinuous solutions of a functional equation* Zeszyty Naukowe UJ, 553, Prace Mat., 22 (1981), 119—123.

[2] M. Kuczma, *Functional Equations in a Single Variable*, Monografie Mat., 46, Polish Scientific Publishers, Warszawa, 1968.

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