UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXVII
1988

Periodic and stationary trajectories of flows and oridinary differential equations

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Abstract. The paper presents some existence theorems for semiflows and ordinary differential equations based on the fixed point index theory.

1. Introduction. In [7] a formula for computing the fixed point index of the translation operator for a semiflow was given. In the present paper the formula will be applied to prove the existence of stationary points for flows and periodic trajectories for ordinary differential equations.

The methods applied in the paper are closely related to the methods used in Ważewski theorem (cf. [10]) and in the Poincare–Bendixson theory (cf. [5], VIII.9.2). Theorem 4.2 is a generalization of a theorem of R. Srzednicki [8] to the case of noncompact set of egress points.

In Part 2 we give preliminary results and definitions. In Part 3 we prove that every semipolyhedron is a BDR (the definitions are given below). It will be used in the sequel. Part 4 contains main results of the paper. An example is given in Part 5.

2. Preliminaries. The following notation will be used. $\boldsymbol{R}^*$ denotes the set of non-negative reals. The norm in $\boldsymbol{R}^n$ is denoted by $||\cdot||$. If X is a metric space and $A \subset X$, then $\bar{A}$, int A, ∂A denote the closure, the interior and the boundary of A, respectively. DA will stand for $\bar{A} \setminus A$. If the Euler characteristic of A is meaningful, it is denoted by $\chi(A)$.

If f: $X \to X$ is a mapping, then Fix$f$ is the set of fixed points of f. Domain and image of f are denoted dom$f$ and im$f$, respectively. We call a mapping π: $\boldsymbol{R}^* \times X \to X$ a semiflow on X if it satisfies the following conditions:

$\pi(0, x) = x$ for every $x \in X$

$\pi(t, \pi(s, x)) = \pi(t+s, x)$ for every $t, s \in \boldsymbol{R}^*$, $x \in X$.

For a fixed $T > 0$

$$\pi_T\colon X \ni x \to \pi_T(x) := \pi(T, x) \in X$$

is the T-translation operator of π. Similarly as in [7] we write $\mathrm{Fix}_T\pi := \mathrm{Fix}\,\pi_T$.

Fix a metric space X. Recall that a subset $A \subset X$ is called a boundary deformation retract (cf. [6]), (BDR for short), if it satisfies the following two conditions:

(2.1) $\bar{A}$ is a compact ANR,

(2.2) if $DA \neq \emptyset$, then there exists an open neighbourhood U of DA and a continuous mapping $r\colon [0, 1] \times \bar{U} \to \bar{U}$ such that

$$\{r(t, x)\colon t \in [0, 1]\} \cap DA = \{r_1(x)\} \text{ for } x \in \bar{U},$$

and

$$r(t, x) = x \Leftrightarrow t = 0 \text{ or } x \in DA\,.$$

If A is a BDR, then the modified Euler characteristic of A is defined by

$$\hat{\chi}(A) := \begin{cases} \chi(\bar{A}) - x(DA) & \text{if } DA \neq \emptyset \\ \chi(A) & \text{if } DA = \emptyset \end{cases}$$

(for the correctness of the definition, compare [7]).

Fix a compact ANR $M \subset X$. For a continuous mapping $f\colon X \to X$ and an open set $U \subset M$, such that $\mathrm{Fix}\,f \cap \partial U = \emptyset$, $i(f, U)$ will stand for the fixed point index of the pair (f, U). If $N \subset M$ is arbitrary $i(f, N)$ will denote $i(f, \mathrm{int}N)$, $(i(f, \emptyset) = 0)$. (For the definition and properties of the fixed point index compare [2] or [4].) Here we recall the following properties of the fixed point index:

(i1) If $U \subset M$ is open, $f\colon X \to X$ is continuous; $\mathrm{Fix}\,f \cap \partial U = \emptyset$ and $i(f, U) \neq 0$ then $\mathrm{Fix}\,f \cap U \neq \emptyset$.

(i2) If $V \subset U \subset M$ are open subsets of M such that $(\bar{U} \setminus V) \cap \mathrm{Fix}\,f = \emptyset$, then $i(f, U) = i(f, V)$

(i3) If $U \subset M$ is open and f_t: is a homotopy such that $\mathrm{Fix}\,f_t \cap \partial U = \phi$ for $t \in [0, 1]$ then $i(f_0, U) = i(f_1, U)$.

Let π be a semiflow on X and let $M \subset X$ be closed. We define the set M^- of egress points of π by

$$M^- := \{x \in M\colon \exists \{\varepsilon_n\}\colon \varepsilon_n > 0,\ \varepsilon_n \to 0,\ \pi(\varepsilon_n, x) \notin M\}\,.$$

Definition 2.1. The set $M \subset X$ will be called *proper for* π if M is a compact ANR, M^- is a BDR and

$$\exists T_0 > 0\ \forall x \in \overline{M^-} \setminus M^-\ \forall t \in [0, T_0]\ \ \pi(t, x) \in M\,.$$

The following result proved in [7] will be basic for the paper.

THEOREM 2.1. *Let π be a semiflow on a metric space X. Assume M is a proper set for π. If there exists a positive number T such that for every $t \in (0, T]$ there are no t-periodic* *trajectories of* π *contained in* M *and intersecting* $\overline{M^-}$ *then*

$$i(\pi_T, M) = \chi(M) - \chi(M^-).$$

Remark 2.1. R. Srzednicki has defined in [8] an index of an isolated invariant set S of a flow π as a difference $\chi(B)-\chi(B^-)$, where $S \subset B$ and B is a block isolating S. (The definitions of isolated invariant sets and isolating blocks may be found in [3].) Theorem 2.1 shows that the index in R. Srzednicki's sense equals to the fixed point index $i(f_t, B)$ for any $t > 0$. (In the case of an isolating block B, $DB^- = \emptyset$.)

3. Boundary deformation retracts. Fix an affine space Z. For $a, b \in Z$ $[a, b]$ will stand for the segment joining the points a, b. Let $s\colon Z \multimap Z$ be a partial mapping.

Definition 3.1. We will say that s satisfies *the property* (W) if for every $a \in \operatorname{dom} s$ $[a, s(a)] \subset \operatorname{dom} s$ and $[a, s(a)] \cap \operatorname{im} s = \{s(a)\}$.

LEMMA 3.1. *Let* $A \subset Z$ *be such that* $\bar{A}$ *is a compact* ANR *and* DA *is closed. If there exists a neighbourhood* U *of* DA *and a continuous mapping* $s\colon U \to DA$ *satisfying* (W), *then* A *is a* BDR.

Proof. We may assume that $U = \bar{U}'$, where U' is an open neighbourhood of DA. Putting

$$r_t\colon \bar{U}' \ni x \to (1-t)x + ts(x) \in \bar{U}' \quad \text{for } t \in [0, 1]$$

we obtain the required in (2.2) homotopy. The proof is finished.

LEMMA 3.2. *Let* K *be a polyhedron and* L *its subpolyhedron. Then* K *admits a triangulation* $\varkappa$ *satisfying the property*

$$\Delta \cap L \text{ is a face of } \Delta \text{ for any } \Delta \in \varkappa. \tag{3.1}$$

Proof. Let $\varkappa_0$ be a triangulation of K and $\alpha_0 \subset \varkappa_0$ a corresponding triangulation of L. Denote by $\varkappa$ the barycentric subdivision of $\varkappa_0$. Fix a q-simplex $\Delta \in \varkappa$. Notice that the set $\{\Delta \cap \Gamma\colon \Gamma \in \varkappa_0\}$ is linearly ordered by the inclusion. Hence we have

$$\Delta \cap L = \Delta \cap \bigcup_{\Gamma \in \alpha_0} \Gamma = \bigcup_{\Gamma \in \alpha_0} \Delta \cap \Gamma = \Delta \cap \Gamma_0,$$

where $\Gamma_0 \in \alpha$ and $\Delta \cap \Gamma_0 = \max\{\Delta \cap \Gamma\colon \Gamma \in \alpha_0\}$. Obviously $\Delta \cap \Gamma_0$ is a face of Δ and the proof is completed.

Fix a simplex Δ with vertices $a_0, a_1, \ldots a_q$ and its nonempty face Γ with vertices $\{a_i\}_{i \in I}$, where $I \subset \{0, 1, \ldots q\}$. Let Γ' be a face of Δ with vertices $\{a_j\}_{j \in J}$, $J := \{0, 1, \ldots q\} \setminus I$. Put $\Delta_\Gamma := \Delta \setminus \Gamma'$. We define the mapping

$$\alpha_\Delta^\Gamma : \Delta_\Gamma \ni a \to \sum_{i \in I} \frac{\alpha_i(a)}{1 - \sum\limits_{j \in J} \alpha_j(a)} a_i \in \Gamma,$$

where $\alpha_i(a)$ denotes the i-th barycentric coordinate of a. It is easy to verify that the mapping is well defined, continuous, satisfies (W) and the property

(3.2) for any face Δ^* of Δ and $\Gamma^* := \Gamma \cap \Delta^*$

$$\alpha^{\Gamma^*}_{\Delta^*} = \alpha^{\Gamma}_{\Delta}|_{\Delta^*_{\Gamma^*}}.$$

Definition 3.2. A subset A of X will be called *a semipolyhedron* if $\bar{A}$ is a polyhedron and DA is its subpolyhedron.

THEOREM 3.1. *Every semipolyhedron is a BDR.*

Proof. Obviously it is enough to restrict ourselves to semipolyhedrons in an affine space. Assume A is such a polyhedron. By Definition 3.2, we can find a polyhedron K and its subpolyhedron L such that $\bar{A} = K$, $DA = L$. Hence $\bar{A}$ is a compact ANR (cf. [1], IV.6.2) and DA is closed.

Choose by Lemma 3.2 a triangulation $\varkappa$ of K satisfying (3.1). Let

$$\mu := \{\Delta \in \varkappa\colon \Delta \cap L = \emptyset\}.$$

The set $M := \bigcup \mu$ is closed. Thus $U := K \setminus M$ is a neighbourhood of L. Let $\Delta \in \varkappa \setminus \mu$. From (3.1) it follows that $\Delta' := \Delta \cap L$ is a non-empty face of Δ. We have also $\Delta_{\Delta'} = \Delta \setminus M$ and

$$U = \bigcup_{\Delta \in \varkappa \setminus \mu} \Delta_{\Delta'}.$$

Define the mapping

$$s := \bigcup_{\Delta \in \varkappa \setminus \mu} \alpha^{\Delta'}_{\Delta} : U \to L.$$

By (3.2) it is well defined. Since all the mappings $\alpha^{\Delta'}_{\Delta}$ satisfy the property (W), so does s. Thus s satisfies the assumptions of Lemma 3.1, by which A is a BDR. The proof is complete.

4. Main results. First we will prove the following

THEOREM 4.1. *Let π be a semiflow on X. Let M be a proper set for π. If $\chi(M) \neq \chi(M^-)$ then there exists a stationary trajectory of π contained in M.*

Proof. If there exists a stationary trajectory of f contained in ∂M, then the thesis is obvious. So we may assume that such a trajectory does not exist. Consequently, for $t > 0$ sufficiently small, there are no t-periodic trajectories of π intersecting ∂M. By Theorem 2.1 and (i1) translation operators π_t have fixed points belonging to $\operatorname{int} M$, i.e. for small $t > 0$ there exist t-periodic trajectories of π contained in M. Now the standard limiting argument shows the existence of a stationary trajectory of π contained in M. The proof is finished.

For a compact set $A \subset \boldsymbol{R}^n$ and $f\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ put

$$\gamma_A(f) := \inf_{x \in A} \|f(x) - x\|, \quad \|f\|_A := \sup_{x \in A} \|f(x)\|.$$

LEMMA 4.1. *Let $f\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ be continuous and let $M \subset \boldsymbol{R}^n$ be a compact set such that $\partial M \cap \operatorname{Fix} f = \emptyset$. Let $g\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ be a continuous mapping such that $\|f-g\|_{\partial M} < \gamma_{\partial M}(f)$. Then $\partial M \cap \operatorname{Fix} g = \emptyset$ and $i(f, M) = i(g, M)$.*

Proof. Obviously $\gamma_{\partial M}(f) > 0$. Put

$$f_t(x) := tf(x)+(1-t)g(x) \text{ for } x \in \boldsymbol{R}^n,\ t \in [0, 1].$$

We have

$$\|f-f_t\|_{\partial M} = |1-t|\,\|f-g\|_{\partial M} \leqslant \|f-g\|_{\partial M}$$

and therefore

$$\gamma_{\partial M}(f) \leqslant \|f-f_t\|_{\partial M}+\gamma_{\partial M}(f_t) \leqslant \|f-g\|_{\partial M}+\gamma_{\partial M}(f_t).$$

Thus

$$\gamma_{\partial M}(f_t) \geqslant \gamma_{\partial M}(f)-\|f-g\|_{\partial M} > 0.$$

The above inequality shows that $\partial M \cap \operatorname{Fix} f_t = \emptyset$ for $t \in [0, 1]$ and in particular, $\partial M \cap \operatorname{Fix} g = \emptyset$. By (i3) $i(f, M) = i(g, M)$. The proof is complete.

The remaining part of this paragraph will concern the existence of periodic solutions of perturbed autonomous differential equations in $\boldsymbol{R}^n$ i.e. the differential equation

$$x' = g(x)+h(t, x) \tag{4.1}$$

where

$$x' = g(x) \tag{4.2}$$

generates a flow in $\boldsymbol{R}^n$.

According to Remark 2.1 the following theorem may be considered as a generalization of Theorem 4.2.11 in [8] to the case when M doesn't necessarily contain an isolated invariant set and M^- is not necessarily compact.

THEOREM 4.2. *Let $\Omega \subset \boldsymbol{R}^n$ be open. Let $g\colon \Omega \to \boldsymbol{R}^n$ be a mapping, such that the equation* (4.2) *generates a flow π on Ω. Let $M \subset \Omega$ be proper for π. Assume that there is a positive number T such that for every $t \in (0, T]$ there are no t-periodic trajectories of π contained in M and intersecting ∂M. If $\chi(M) \neq \chi(M^-)$, then for every U a neighbourhood of*

$$\operatorname{Fix}_T^M \pi := \operatorname{Fix}_T \pi \cap \{x \in M\colon \forall t \in \boldsymbol{R}^*\ \pi(t, x) \in M\}$$

there exists a $\delta > 0$ such that for every continuous mapping $h\colon \boldsymbol{R} \times \Omega \ni (t, x) \to h(t, x) \in \boldsymbol{R}^n$, T-periodic in t and such that $\|h\|_{\boldsymbol{R} \times \Omega} < \delta$ the equation (4.1) *admits a T-periodic trajectory contained in U.*

Proof. Obviously we may assume that $U \subset M$. Choose compact neighbourhoods N_1, N_2 of M such that $M \subset N_1 \subset \operatorname{int} N_2 \subset \Omega$. Let $k\colon \Omega \to [0, 1]$ be a continuous mapping such that $k(x) = 1$ for $x \in N_1$ and $k(x) = 0$ for $x \notin N_2$. Then the equation

$$x' = g(x)+k(x)h(t, x) \tag{4.3}$$

has the same solutions as (4.1) in N_1 and its solutions exist for all t. Obviously we may consider (4.3) instead of (4.1). First assume that the initial value problems for (4.3) have unique solutions and let $y(\cdot, x)$ be the solution of (4.3) satisfying $y(0, x) = x$. Set $w\colon \Omega \ni x \to y(T, x) \in \Omega$. The continuous dependence on the right side of the equation (4.2) and compactness of the set $\mathrm{Fix}_T^M \pi \times [0, T]$ permits us to choose a $\delta_0 > 0$ and a neighbourhood V of Fix_T^M such that

$$(4.4) \qquad ||h||_{\boldsymbol{R}\times\Omega} < \delta_0 \Rightarrow y(t, x) \in U \text{ for all } (t, x) \in [0, T]\times V.$$

Similarly, we may choose *a* $\delta > 0$, $\delta \leqslant \delta_0$ such that

$$||h||_{\boldsymbol{R}\times\Omega} < \delta \Rightarrow ||\pi_T - w||_{\partial V} < \nu_{\partial V}(\pi_T).$$

Now by Theorem 2.2, the property (i2) and Lemma 4.1 we get

$$0 \neq \chi(M) - \hat{\chi}(M^-) = i(\pi_T, M) = i(\pi_T, V) = i(w, V).$$

From (i1) it follows that there exists a fixed point $\bar{x}$ of w in V, hence $y(t, \bar{x})$ is T-periodic and by (4.4) it is contained in U.

The general case is obtained by uniform approximation of h on $[0, T]\times M$ and a standard limiting argument. The proof is complete.

Definition 4.1. A mapping $y\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ satisfies *the property* (P) iff

$$(4.5) \qquad \forall P > 0\, \exists K > 0\, ||x|| > K \Rightarrow ||y(x) - x|| \geqslant P.$$

THEOREM 4.3. *Let for a L-Lipschitz mapping* $g\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ *the equation* (4.2) *generates a flow* π *on* $\boldsymbol{R}^n$ *such that* π_T *satisfies* (P). *Assume that there exists a proper set* M *for* π *and a* $T > 0$ *such that for* $t \in (0, T]$ *all* t*-periodic trajectories of* π *are contained in* $\mathrm{int}\, M$ *and* $\chi(M) \neq \hat{\chi}(M^-)$. *If a continuous mapping* $h\colon \boldsymbol{R}\times\boldsymbol{R}^n \ni (t, x) \to h(t, x) \in \boldsymbol{R}^n$, *T-periodic in* t *satisfies the inequality*

$$(4.6) \qquad ||h(t, x)|| \leqslant N + \mu||x||, \quad t \in \boldsymbol{R},\ x \in \boldsymbol{R}^n$$

for a constant $N > 0$ *and a suitable small* $\mu = \mu(N)$ *then the equation* (4.1) *admits a T-periodic trajectory.*

Proof. By Lipschitzity of g and (4.6) the solutions of (4.1) are estimated by solutions of a linear equation and consequently exist for all t (cf. [9], Theorem 23.1).

Fix $P > TNe_{LT}$ and choose K so large that $||\pi_T(x) - x|| \geqslant P$ for $||x|| \geqslant K$ and $M \subset B := \{x\colon ||x|| \leqslant K\}$. For $x \in \boldsymbol{R}^n$ let A_x be the set of all solutions y of (4.1) satisfying the initial condition $y(0) = x$. Put

$$R := \max\{y(t)\colon t \in [0, T],\ x \in B,\ y \in A_x\}.$$

We have for any $x \in B$ and $y \in A_x$:

$$\begin{aligned} ||y(T) - \pi(T, x)|| &\leqslant ||\int_0^T [g(y(t)) + h(t, y(t)) - g(\pi(t, x))]\, dt \leqslant \\ &\leqslant \int_0^T ||h(t, y(t))||\, dt + \int_0^T ||g((y(t)) - g(\pi(t, x))||\, dt \leqslant \\ &\leqslant T(N + \mu R) + \int_0^T L||y(t) - \pi(t, x)||\, dt. \end{aligned}$$

Applying Gronwall inequality and passing to supremum we get

$$\sup_{y\in A_x}\|y(T,x)-\pi(T,x)\|\leqslant T(N+\mu R)e^{LT}.$$

Assume for a moment that the initial value problems for (4.1) have unique solutions and let

$$w\colon \boldsymbol{R}^n\ni x\to y(T)\in \boldsymbol{R}^n,\ \text{where } y\in A_x.$$

Since for μ small enough $T(N+\mu R)e^{LT}<P$, from Lemma 4.1, the property (i2) and Theorem 2.1 we obtain

$$i(w,B)=i(\pi_T,B)=i(\pi_T,M)=\chi(M)-\hat{\chi}(M^-)\neq 0.$$

By (i1) there exists a fixed point of w and consequently a T-periodic trajectory of (4.1). The general case is obtained by uniform approximation of h on $[0,T]\times B$ and a standard limiting argument. The proof is finished.

Finally we will show how to obtain a simpler proof of a version of Poincare–Bendixson Theorem ([5], VII.9.2).

THEOREM 4.4. *Let $E\subset \boldsymbol{R}^2$ be open and simply connected and let for a continuous mapping $g\colon E\to \boldsymbol{R}^2$* (4.2) *generate a flow π on E. Let C be a Jordan curve in E of class C^1, surrounding a compact $M\subset E$. Assume that $g(y)\neq 0$ for $y\in C$ and that $g(y)$ is tangent to C at only a finite number of points $y_1, y_2, \dots y_n\in C$. Let n^e, n^h be the number of points, which trajectories are internally, externally tangent to C. Then for small t*:

$$2i(\pi_t,M)=2+n^e-n^h. \tag{4.7}$$

Proof. Observe that M, as homeomorphic to a 2-dimensional ball is obviously an ANR and $\chi(M)=1$. If $n^e=n^h=0$ then $M^-=\emptyset$ or $M^-=C$ and by Theorem 3.1, it is a BDR. In the other case M^- is a finite sum of open, closed or one-sided open segments with disjoint closures. So again by Theorem 3.1, M^- is a BDR. Obviously the set $\overline{M^-}\setminus M^-$ is finite. Hence we can choose a $T_0>0$ such that for $x\in \overline{M^-}\setminus M^-$ and all $t\in[0,T_0]$, $\pi(t,x)\in M$. So we have proved that M is proper for π. In order to apply Theorem 2.1 it suffices to verify that

$$\hat{\chi}(M^-)=\frac{n^h-n^e}{2}. \tag{4.8}$$

If $M^-=\emptyset$ or $M^-=C$, then $\hat{\chi}(M^-)=0$ and (4.8) is satisfied. In the remaining case, as we observed,

$$M^-=\bigcup_{i=1}^{k}A_i,$$

where A_i, $i=1,2,\dots k$ are segments of C (open, closed or one-sided closed) with disjoint closures. In all cases we have

$$\hat{\chi}(A_i)=\frac{a_i^h-a_i^e}{2},\qquad i=1,2,\dots k, \tag{4.9}$$

where a_i^h (a_i^e) denote the number of externally (internally) tangent vectors of g in $\overline{A_i}$. Summing up (4.9) for $i = 1, 2, \dots k$ we obtain (4.8), which by Theorem 2.1 completes the proof.

Remark 4.1. The Poincare–Bendixson theorem differs from Theorem 4.4 only in the fact that the fixed point index appearing in (4.7) is replaced in Poincare–Bendixson theorem by another index (in fact by the degree of g). But both theorems imply the following.

COROLLARY 4.2. *By the assumptions of Theorem* 4.4, *if* $2+n^e-n^h \neq 0$ *then* (4.2) *has a stationary trajectory.*

5. Example. Some examples of the above theorems in the case when M^- is simply a compact ANR may be found in [8]. Here is another example. Let $g\colon \boldsymbol{R}^2 \to R^2$ be defined by

$$g(x, y) := \begin{cases} (0,0) & \text{for } (x,y) = (0,0) \\ \left(x\dfrac{x^2-2y^2}{x^2+y^2}, y\dfrac{2x^2-y^2}{x^2+y^2}\right) & \text{otherwise}. \end{cases}$$

It may be verified that g is a Lipschitz function generating a flow π on $\boldsymbol{R}^2$ given by

$$\pi(t, (x, y)) := \begin{cases} (0,0) & \text{for } (x,y) = (0,0) \\ (xe^t p(t, x, y), ye^{2t} p(t, x, y)) & \text{otherwise} \end{cases} \tag{5.1}$$

where $p(t, x, y) := (x^2+y^2)^{3/2}(x^2+e^{2t}y^2)^{-3/2}$ (see Fig. 1).

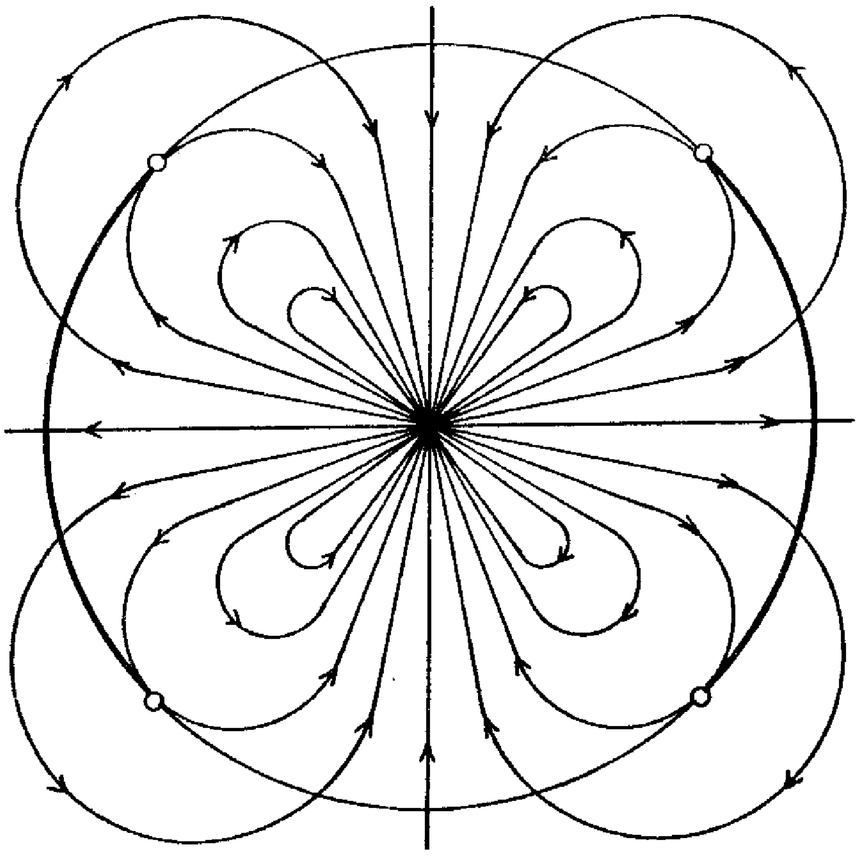

Fig. 1.