UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXVII

1988

Solution of the generalized translation equation on some structures

IRENA ADAMASZEK

The translation equation

$$F(F(x, a), b) = F(x, g(a, b)) \tag{T_2}$$

belongs to the most known and important functional equations and has essential applications in geometry, iteration theory, probability, nomography and physics. It has been a subject of investigations for different structures: groups, semigroups, groupoids, categories (see [1]).

In 1979 Ł. Piechowicz and S. Serafin [4] gave necessary and sufficient conditions for a mapping $F\colon M \times S \to M$ (with M, S — arbitrary nonempty sets) to be a solution of equation (T_2) in the case where g is a binary law in S and

(1) $\operatorname{card} g(S \times S) = 1$

or

(2) $g(a, b) = b$ for all $(a, b) \in S \times S$

or

(3) $g(a, b) = a$ for all $(a, b) \in S \times S$.

Obviously, the pair (S, g) forms a semigroup in each of the above three cases.

In this paper we present the general solution of a generalized translation equation

$$F(\dots F(F(x, a_1), a_2) \dots, a_n) = F(x, g(a_1, \dots, a_n)) \tag{T_n}$$

in the case where $n \geqslant 2$ is a positive integer, g maps S^n into S and either

(4) $\operatorname{card} g(S^n) = 1$

or

(5) $g(a_1, \dots, a_n) = a_k$ for all $(a_1, \dots, a_n) \in S^n$

where $k \in \{1, \dots, n\}$ is arbitrarily fixed.

The above mentioned results of Ł. Piechowicz and S. Serafin in [4] are particular cases of our Theorems 1, 2, 3′ ($n = 2$).

Definition 1. (cf. [3]). Let $n \geqslant 2$ be a positive integer. An algebraic system (G, h) is said to be *an n-semigroup* provided the n-ary operation $h\colon G^n \to G$ satisfies the following condition:

$$h(h(x_1, \ldots, x_n), x_{n+1}, \ldots, x_{2n-1}) = \\ = h(x_1, \ldots, x_i, h(x_{i+1}, \ldots, x_{i+n}), x_{i+n+1}, \ldots, x_{2n-1})$$

for all $x_1, x_2, \ldots, x_{2n-1} \in G$ and $i = 1, 2, \ldots, n-1$.

Some n-semigroups (in particular, some n-groups) have applications in geometry (see [2]).

If $g\colon S^n \to S$ satisfies condition (4) or (5) for $k = 1$ or $k = n$ then, obviously, the system (S, g) is an n-semigroup.

Let Φ be the family of all functions $f\colon M \to M$ and let $\varphi\colon \Phi^n \to \Phi$ be defined by the formula:

$$\varphi(f_1, \ldots, f_n) = f_n \circ \ldots \circ f_2 \circ f_1 \quad \text{for all} \quad (f_1, \ldots, f_n) \in \Phi^n;$$

here the symbol "$\circ$" denotes the superposition operation. Clearly, the system (Φ, φ) is an n-semigroup.

Observe that for any non-empty family $\Psi \subset \Phi$ such that

(6) $\quad \varphi(\Psi^n) \subset \Psi$ and $\operatorname{card} \varphi(\Psi^n) = 1$

or

(7) $\quad \varphi(f_1, \ldots, f_n) = f_n$ for all $(f_1, \ldots, f_n) \in \Psi^n$

or

(8) $\quad \varphi(f_1, \ldots, f_n) = f_1$ for all $(f_1, \ldots, f_n) \in \Psi^n$

the system (Ψ, φ) yields an n-semigroup (an n-subsemigroup of (Φ, φ)).

Theorems 1, 2, 3 of our paper allow to verify whether the system (Ψ, φ) yields one of the three above mentioned n-semigroups.

In the sequel, if f maps a given set into itself and n is a positive integer, then f^n will always denote the n-th iterate of f.

Definition 2. A family $\{M_i\}_{i \in I}$ of nonempty subsets of a set M is called to be *a partition* of M provided $\bigcup_{i \in I} M_i = M$ and $M_i \cap M_j = \phi$ for all $i \neq j$, $i, j \in I$.

Now, suppose that $g\colon S^n \to S$ satisfies condition (4). Then equation (T_n) may be written in the form

$$F(\ldots F(F(x, a_1), a_2) \ldots, a_n) = F(x, e) \tag{T}$$

where $e = g(a_1, \ldots, a_n)$ for all $a_1, \ldots, a_n \in S$.

THEOREM 1. *A mapping $F\colon M \times S \to M$ is a solution of equation* (T) *if and only if there exist a partition $\{M_i\}_{i \in I}$ of M, a family $\{P_i\}_{i \in I}$ of nonempty subsets of M and a family $\{f_i\}_{i \in I}$ of mappings such that the following conditions are satisfied*:

(1.1) *every set M_k contains exactly one set P_i*;

(1.2) $f_i\colon M_i \times S \to P_i$ *for all* $i \in I$;

$$(1.3)\quad \begin{cases} \textit{if } f_{i_1}\colon M_{i_1} \times S \to P_{i_1} \subset M_{i_2} \\ \quad f_{i_2}\colon M_{i_2} \times S \to P_{i_2} \subset M_{i_3} \\ \quad \dots\dots\dots\dots\dots \\ \quad f_{i_{n-1}}\colon M_{i_{n-1}} \times S \to P_{i_{n-1}} \subset M_{i_n}, \\ \textit{then } M_{i_n} = M_{i_1} \textit{ for all } i_1, i_2, \dots, i_n \in I; \end{cases}$$

$$(1.4)\quad \begin{cases} \textit{for every } i_1 \in I \textit{ there exists exactly one element } x_{i_1} \in P_{i_1} \textit{ such that } f_{i_1}(x, e) = x_{i_1} \\ \textit{for all } x \in M_{i_1} \textit{ and } f_{i_1}(x, a) = x_{i_1} \textit{ for all } a \in S, x \in f_{i_{n-1}}\big(\dots f_{i_2}\big(f_{i_1}(M_{i_1} \times S) \times S\big) \times S \dots \\ \dots \times S\big) \textit{ where } i_2, \dots, i_{n-1} \in I \textit{ are arbitrary, provided} \\ f_{i_j}\colon M_{i_j} \times S \to M_{i_{j+1}} \textit{ for } 1 \leqslant j \leqslant n-1; \end{cases}$$

(1.5) $F(x, a) = f_i(x, a)$ *for all* $x \in M_i$, $a \in S$, $i \in I$.

Proof. Suppose that $F\colon M \times S \to M$ is a solution of equation (T) and define a function $f\colon M \to M$ by the formula:

$$f(x) := F(x, e) \quad \text{for every } x \in M.$$

It is not hard to check that if $a_1 = a_2 = \dots = a_n = e$, then equation (T) implies

$$(1.6)\qquad f(x) = f^n(x) \quad \text{for every } x \in M.$$

Let $I := f(M)$ and $M_i := f^{-1}(\{i\})$ for every $i \in I$. Then the family $\{M_i\}_{i \in I}$ is a partition of M.

Now, we define a function $h\colon M \to M$ by the formula:

$$h(x) := f^{n-1}(x) \quad \text{for all } x \in M.$$

Since $h(x) = f(f^{n-2}(x))$ for all $x \in M$ we have $h(M) \subset f(M)$. Moreover $f(x) = f^n(x) = h(f(x))$ for all $x \in M$; therefore $f(M) \subset h(M)$. Hence $h(M) = f(M) = I$.

Let $N_i := h^{-1}(\{i\})$ for every $i \in I$. The family $\{N_i\}_{i \in I}$ is a partition of M. In addition, $f(x) = f(h(x)) = f(i)$ for all $x \in N_i$, $i \in I$. So, $f\colon N_i \to \{f(i)\}$ for every $i \in I$ and, obviously, $f(i) \in I$, $i \in I$. Therefore, for every $i \in I$, there exists a $k \in I$ such that

$$(1.7)\qquad N_i \subset M_k.$$

Moreover $h(x) = f^{n-2}(f(x)) = f^{n-2}(k)$ for all $x \in M_k$, $k \in I$. Hence $h\colon M_k \to \{f^{n-2}(k)\}$ for all $k \in I$ and, obviously, $f^{n-2}(k) \in I$, $k \in I$. Therefore, for every $k \in I$, there exists a $j \in I$ such that

$$(1.8)\qquad M_k \subset N_j.$$

From conditions (1.7) and (1.8) it follows that for every $i \in I$ there exist $k \in I$ and $j \in I$ such that $N_i \subset M_k \subset N_j$. Since $N_i \cap N_j = \emptyset$ for $i \neq j$, we get $i = j$. Thus, for every $i \in I$, there exists exactly one $k \in I$ such that

$$(1.9)\qquad N_i = M_k.$$

From the definition of h it follows that for all $i \in I$ and $(x, a) \in M \times S$ the following equalities hold

$$i = F(x, e) = F(\dots F(F(x, a), e)\cdots, e) = h(F(x, a)).$$

Hence $F(x, a) \in N_i$ for all $(x, a) \in M_i \times S$, $i \in I$. Therefore, $F\colon M_i \times S \to N_i$ for all $i \in I$. Put $P_i := F(M_i \times S)$ for all $i \in I$. Obviously, $P_i \subset N_i$ for every $i \in I$. From condition (1.9) it follows that the family $\{P_i\}_{i \in I}$ satisfies condition (1.1).

Since $F(x, e) = f(x) = i$ for all $x \in M_i$, $i \in I$, we have $i \in P_i$ for every $i \in I$.

Put $f_i := F|_{M_i \times S}$, $i \in I$. Let $i_1, \dots, i_{n-1} \in I$ be such that

$$\begin{aligned} &f_{i_1}\colon M_{i_1} \times S \to P_{i_1} \subset M_{i_2} \\ &f_{i_2}\colon M_{i_2} \times S \to P_{i_2} \subset M_{i_3} \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &f_{i_{n-1}}\colon M_{i_{n-1}} \times S \to P_{i_{n-1}} \subset M_{i_n}. \end{aligned}$$

Then $f_{i_{n-1}}(\cdots f_{i_2}(f_{i_1}(M_{i_1} \times S) \times S) \cdots \times S) \subset P_{i_{n-1}} \subset M_{i_n}$.
Moreover

$$\begin{aligned} &f(f_{i_{n-1}}(\cdots f_{i_2}(f_{i_1}(M_{i_1} \times S) \times S) \cdots \times S)) = \\ &= f(F(\cdots F(F(M_{i_1} \times S) \times S) \cdots \times S)) = \\ &= F(F(\cdots F(F(M_{i_1} \times S) \times S) \cdots \times S) \times \{e\}) = \\ &= F(M_{i_1} \times \{e\}) = f(M_{i_1}) = \{i_1\}. \end{aligned}$$

Hence

$$f_{i_{n-1}}(\cdots f_{i_2}(f_{i_1}(M_{i_1} \times S) \times S) \cdots \times S) \subset M_{i_1}.$$

Therefore $M_{i_n} = M_{i_1}$. Consequently, the family $\{f_i\}_{i \in I}$ satisfies condition (1.3).

We shall show that condition (1.4) is satisfied, too. Put $x_i := i$ for every $i \in I$. Note that

$$f_{i_1}(x, e) = F|_{M_{i_1} \times S}(x, e) = f|_{M_{i_1}}(x) = i_1 = x_{i_1}$$

for all $x \in M_{i_1}$, $i_1 \in I$. On the other hand, if $x \in f_{i_{n-1}}(\cdots f_{i_2}(f_{i_1}(M_{i_1} \times S) \times S) \cdots \times S)$, then there exist a $y \in M_{i_1}$ and $a_1, \dots, a_{n-1} \in S$ such that $x = f_{i_{n-1}}(\cdots f_{i_2}(f_{i_1}(y, a_1), a_2) \cdots, a_{n-1})$ whence

$$\begin{aligned} f_{i_1}(x, a) &= f_{i_1}(f_{i_{n-1}}(\cdots f_{i_2}(f_{i_1}(y, a_1), a_2) \cdots, a_{n-1}), a) = \\ &= F(F(\cdots F(F(y, a_1), a_2) \cdots, a_{n-1}), a) = \\ &= F(y, a) = f(y) = i_1 = x_{i_1} \end{aligned}$$

for all $a \in S$. This proves that condition (1.4) holds true.

Now, suppose that there exist: a partition $\{M_i\}_{i \in I}$ of M, a family $\{P_i\}_{i \in I}$ of nonempty subsets of M and a family $\{f_i\}_{i \in I}$ of mappings such that conditions (1.1), (1.2), (1.3) and (1.4) are satisfied and a mapping $F\colon M \times S \to M$ satisfies condition (1.5). Then for every $i_1 \in I$ there exist $i_2, \dots, i_{n-1} \in I$ such that

$$\begin{aligned} &F(F(\cdots F(F(x, a_1), a_2) \dots, a_{n-1}), a_n) = \\ &= F(f_{i_{n-1}}(\cdots f_{i_2}(f_{i_1}(x, a_1), a_2) \cdots, a_{n-1}), a_n) = \\ &= f_{i_1}(f_{i_{n-1}}(\cdots f_{i_2}(f_{i_1}(x, a_1), a_2) \cdots, a_{n-1}), a_n) = \\ &= x_{i_1} = f_{i_1}(x, e) = F(x, e) \end{aligned}$$

for all $x \in M_{i_1}$, $a_1, \ldots, a_n \in S$. Therefore, F is a solution of equation (T). This completes our proof.

Now, suppose that $g\colon S^n \to S$ satisfies condition (5) for $k = n$ (n-semigroup of left "units"). Then $g(a_1, \ldots, a_n) = a_n$ for all $a_1, \ldots, a_n \in S$ and equation (T_n) may be written in the form

$$\text{(T')} \qquad F(\cdots F(F(x, a_1), a_2)\cdots, a_n) = F(x, a_n)\,.$$

Theorem 2. *A mapping $F\colon M \times S \to M$ is a solution of equation* (T′) *if and only if there exist a partition $\{M_i\}_{i \in I}$ of M, a family J of functions $f\colon M \to M$ and a function $h\colon S \to J$ such that the following conditions are satisfied*:

(2.1) *if $f \in J$, then $f^{n-1}(M_i) \subset M_i$ and* $\operatorname{card} f(M_i) = 1$ *for every $i \in I$*;

(2.2) *$f_1 \circ f_2 = f_1 \circ f_3$ for all $f_1, f_2, f_3 \in J$*;

(2.3) *$F(x, a) = \big(h(a)\big)(x)$ for all $(x, a) \in M \times S$.*

Proof. Suppose that $F\colon M \times S \to M$ is a solution of equation (T′). Let J be a family of all functions $f_a\colon M \to M$ given by the formula:

$$\text{(2.4)} \qquad f_a(x) := F(x, a) \quad \text{for all } x \in M,\ a \in S\,.$$

Let $e \in S$ be a fixed element and $I := f_e^{n-1}(M)$. We define a family $\{M_i\}_{i \in I}$ of non-empty subsets of the set M in the following way:

$$M_i := (f_e^{n-1})^{-1}(\{i\}) \qquad \text{for every } i \in I\,.$$

Of course, $\{M_i\}_{i \in I}$ is a partition of M.

Let $a \in S$ be arbitrary. Since $f_a(x) = f_a(f_e^{n-1}(x))$ for all $x \in M$ and $f_e^{n-1}(x) = i$ for all $x \in M_i$, $i \in I$, we obtain

$$\text{(2.5)} \qquad f_a(x) = f_a(f_e^{n-1}(x)) = f_a(i) \quad \text{for all } x \in M_i,\ i \in I\,.$$

Hence

$$\operatorname{card} f_a(M_i) = 1 \qquad \text{for all } a \in S,\ i \in I\,.$$

Equalities (2.5) imply, in particular, that $f_e(x) = f_e(i)$ for all $x \in M_i$, $i \in I$. Therefore $f_e^{n-1}(x) = f_e^{n-1}(i)$ for all $x \in M_i$, $i \in I$. Since $f_e^{n-1}(x) = i$ for all $x \in M_i$, $i \in I$, we get

$$\text{(2.6)} \qquad i = f_e^{n-1}(i) \quad \text{for all } i \in I\,.$$

Moreover, equalities (2.5) imply that $f_a^{n-1}(M_i) = \{f_a^{n-1}(i)\}$ for all $i \in I$, $a \in S$. In addition,

$$f_e^{n-1}(f_a^{n-1}(i)) = f_e^{n-2}(f_e(f_a^{n-1}(i))) = f_e^{n-2}(f_e(i)) = f_e^{n-1}(i)$$

for all $i \in I$, $a \in S$. Hence and from (2.6) it follows that $f_e^{n-1}\big(f_a^{n-1}(i)\big) = i$ for all $i \in I$, $a \in S$. This means that $f_a^{n-1}(M_i) \subset M_i$ for all $i \in I$, $a \in S$ and condition (2.1) is satisfied.

Now, we shall prove that the family J satisfies condition (2.2). We notice that

$$\begin{aligned} F(F(x, a_n), a_{n+1}) &= F\big(F(\cdots F(F(x, a_1), a_2)\cdots, a_n), a_{n+1}\big) = \\ &= F(F(x, a_1), a_{n+1}) \end{aligned}$$

for all $x \in M$, $a_1, \ldots, a_n, a_{n+1} \in S$. Hence

$$F(F(x,a),c) = F(F(x,b),c) \quad \text{for all } x \in M,\ a, b,\ c \in S.$$

Therefore $f_c \circ f_a = f_c \circ f_b$ for all $f_a, f_b, f_c \in J$.

Let h: $S \to J$ be given by the formula:

$$h(a) := f_a \quad \text{for every } a \in S.$$

Then, obviously, condition (2.3) is satisfied.

Now, suppose that there exist a partition $\{M_i\}_{i \in I}$ of M, a family J of functions f: $M \to M$ and a function h: $S \to J$ such that conditions (2.1) and (2.2) are satisfied and that a mapping F: $M \times S \to M$ satisfies condition (2.3). Put $f_a := h(a)$ for every $a \in S$. Let $i \in I$ be arbitrary, $x \in M_i$, $a_1, \ldots, a_n \in S$. Then conditions (2.3), (2.2) and (2.1) imply the following equalities:

$$F(\cdots F(F(x,a_1),a_2)\cdots,a_n) = f_{a_n} \circ \cdots \circ f_{a_2} \circ f_{a_1}(x) =$$

$$= f_{a_n} \circ \cdots \circ f_{a_2} \circ f_{a_n}(x) = \cdots = f_{a_n} \circ f_{a_n}^{n-1}(x) = f_{a_n}(y),$$

where $y \in M_i$. Since $\operatorname{card} f_a(M_i) = 1$ for all $i \in I$, $a \in S$, then, obviously, $f_{a_n}(y) = f_{a_n}(x) = F(x, a_n)$. This proves that the mapping F satisfies equation (T′) which completes our proof.

We notice that for $n = 2$ condition (2.2) is redundant (it is seen that (2.2) is a consequence of (2.1) in the case where $n = 2$).

Remark 1. Let id_M: $M \to M$ be the identity mapping on M. If F: $M \times S \to M$ is a solution of equation (T′), J is a family of all functions f_a: $M \to M$ given by the formula (2.4) and there exists a $c \in S$ such that $F(x, c) = x$ for all $x \in M$, then J contains the function id_M. Since J satisfies condition (2.2), $f_a = \mathrm{id}_M \circ f_a = \mathrm{id}_M \circ \mathrm{id}_M = \mathrm{id}_M$ for all $a \in S$, whence $F(x,a) = x$ for all $(x,a) \in M \times S$. Similarly, if a family J of functions f: $M \to M$ satisfies condition (2.2) and $\mathrm{id}_M \in J$, then $J = \{\mathrm{id}_M\}$. Moreover, if this family J, a partition $\{M_i\}_{i \in I}$ of M, a function h: $S \to J$ and a mapping F: $M \times S \to M$ satisfy conditions (2.1) and (2.3), then $I = M$, $M_i = \{i\}$ for every $i \in I$ and the solution F of equation (T′) has the following form: $F(x,a) = x$ for all $(x,a) \in M \times S$.

Remark 2. With almost no changes in the proof one may show that Theorem 2 remains true in the case where condition (2.1) is replaced by the following conditions:

(i) if $f \in J$, then $\operatorname{card} f(M_i) = 1$ for every $i \in I$;
(ii) there exists a function $f_0 \in J$ such that $f_0^{n-1}(M_i) \subset M_i$ for every $i \in I$;
(iii) there exists an $e \in S$ such that $h(e) = f_0$.

Now, suppose that g: $S^n \to S$ satisfies condition (5) for $k = 1$ (n-semigroup of right "units"). Then $g(a_1, \ldots, a_n) = a_1$ for all $a_1, \ldots, a_n \in S$ and equation (T_n) may be written in the form

$$(\mathrm{T}'') \qquad F(\cdots F(F(x,a_1),a_2)\cdots,a_n) = F(x,a_1).$$

THEOREM 3. *A mapping $F\colon M\times S\to M$ is a solution of equation* (T″) *if and only if there exist a family J of functions $f\colon M\to M$ and a function $h\colon S\to J$ such that the following conditions are satisfied*:

(3.1) *if $f\in J$, then $f^{n-1}(x)=x$ for all $x\in f(M)$*;

(3.2) *$f_2\circ f_1=f_3\circ f_1$ for all $f_1, f_2, f_3\in J$*;

(3.3) *$F(x,a)=\big(h(a)\big)(x)$ for all $(x,a)\in M\times S$.*

Proof. Suppose that $F\colon M\times S\to M$ is a solution of equation (T″). Let J be a family of all functions $f_a\colon M\to M$ given by the formula:

$$f_a(x):=F(x,a)\quad \text{for all } x\in M,\ a\in S. \tag{3.4}$$

Fix an $a\in S$ arbitrarily. One has

$$f_a(x)=f_a^n(x)=f_a^{n-1}(f_a(x))\quad \text{for all } x\in M.$$

Therefore $f_a^{n-1}(x)=x$ for all $x\in f_a(M)$. Consequently, the family J satisfies condition (3.1).

We shall show that the family J satisfies condition (3.2), too. Remark that

$$F\big(F(x,a_1),a_2\big)=F\big(F(\cdots F\big(F(x,a_1),a_2\big)\cdots,a_n\big),a_{n+1}\big)=$$
$$=F\big(F(x,a_1),a_{n+1}\big)$$

for all $x\in M$, $a_1,a_2,\ldots,a_{n+1}\in S$. Hence

$$F\big(F(x,a),b\big)=F\big(F(x,a),c\big)\quad \text{for all } x\in M,\ a,\ b,\ c\in S.$$

Therefore $f_b\circ f_a=f_c\circ f_a$ for all $f_a, f_b, f_c\in J$.

Let $h\colon S\to J$ be given by the formula:

$$h(a):=f_a\quad \text{for every } a\in S.$$

Then, obviously, condition (3.3) is also satisfied.

Now, suppose that there exist a family J of functions $f\colon M\to M$ and a function $h\colon S\to J$ such that conditions (3.1) and (3.2) are satisfied and that the mapping $F\colon M\times S\to M$ satisfies condition (3.3). Let $f_a:=h(a)$ for every $a\in S$. From conditions (3.3), (3.2) and (3.1) it follows that

$$F\big(\cdots F\big(F(x,a_1),a_2\big)\cdots,a_n\big)=f_{a_n}\circ\cdots\circ f_{a_2}\circ f_{a_1}(x)=f_{a_n}\circ\cdots\circ f_{a_3}\circ f_{a_1}\circ f_{a_1}(x)$$
$$=\cdots=f_{a_1}^n(x)=f_{a_1}^{n-1}(f_{a_1}(x))=f_{a_1}(x)=F(x,a_1)$$

for all $x\in M$, $a_1,\ldots,a_n\in S$. This proves that the mapping F satisfies equation (T″), which completes our proof.

Remark 3. An observation analogous to that mentioned in Remark 1 remains valid with regard to equation (T″); obviously, one has to replace conditions (2.1), (2.2), (2.3) and (2.4) by (3.1), (3.2), (3.3) and (3.4), respectively.

Noteworthy is also the following alternative version of Theorem 3.

THEOREM 3′. *A mapping* $F\colon M\times S\to M$ *is a solution of equation* (T″) *if and only if there exist a nonempty subset* V *of* M, *a family* J *of functions* $f\colon M\to M$ *and a function* $h\colon S\to J$ *such that the following conditions are satisfied*:

(3.1)′ *if* $f\in J$, *then* $f(M)=V$ *and* $f^{n-1}(x)=x$ *for all* $x\in V$;

(3.2) $f_2\circ f_1=f_3\circ f_1$ *for all* $f_1, f_2, f_3\in J$;

(3.3) $F(x,a)=\big(h(a)\big)(x)$ *for all* $(x,a)\in M\times S$.

Proof. Sufficiency results immediately from the obvious observation that (3.1)′ implies (3.1). To prove the necessity, suppose that $F\colon M\times S\to M$ is a solution of equation (T″) and define the family J as the collection of all selfmappings of M given by the formula (3.4). Put $V:=\bigcup\limits_{a\in S}f_a(M)$. Let $a\in S$ be arbitrary and let $y\in V$. Then there exist a $b\in S$ and an $x\in M$ such that $y=f_b(x)$. Moreover,

$$y=f_b(x)=F(x,b)=F\big(F(\cdots F\big(F(x,b),b\big)\cdots,b\big),a\big)=f_a(f_b^{n-1}(x)).$$

Hence $y\in f_a(M)$ and, therefore, $V=f_a(M)$ for all $a\in S$. In addition,

$$f_a(x)=f_a^n(x)=f_a^{n-1}\big(f_a(x)\big)\quad \text{for all } x\in M,\ a\in S.$$

Thus $f_a^{n-1}(x)=x$ for all $x\in V$. Consequently, the family J satisfies condition (3.1)′.
The rest of the proof is literally the same as that of Theorem 3.

Remark 4. Note that for $n=2$ condition (3.2) is redundant (it is seen that (3.2) is a consequence of (3.1)′ in that case).

COROLLARY 1. *Let* Φ *denote the family of all functions* $f\colon M\to M$. *Let* Ψ *be a nonempty subfamily of* Φ *indexed by elements of a set* S, *i.e.* $\Psi=\{f_a\in\Phi\colon a\in S\}$. *Put*

$$F(x,a):=f_a(x),\ (x,a)\in M\times S;$$

thus, F *is a mapping from* $M\times S$ *into* M. *Clearly,* Ψ *satisfies* (6), (7) *or* (8) *if and only if* F *is a solution of equation* (T), (T′) *or* (T″), *respectively. On account of our remarks before* Definition 1 *in each of the above three cases the pair* (Ψ,φ) *forms an* n*-subsemigroup of* (Φ,φ).

Now, suppose that $g\colon S^n\to S$ satisfies condition (5) for a positive integer $k\in\{2,\dots,n-1\}$. Then $g(a_1,\dots,a_n)=a_k$ for all $a_1,\dots,a_n\in S$ and equation (T_n) may be written in the form

$$F(\cdots F(F(x,a_1),a_2)\cdots,a_n)=F(x,a_k). \tag{T‴}$$

THEOREM 4. *A mapping* $F\colon M\times S\to M$ *is a solution of equation* (T‴) *if and only if there exists a function* $f\colon M\to M$ *such that* $F(x,a)=f(x)$ *for all* $(x,a)\in M\times S$ *and* $f^n(x)=f(x)$ *for all* $x\in M$.

Proof. Suppose that a mapping $F\colon M \times S \to M$ is a solution of equation (T‴). Let J denote a family of all functions $f_a\colon M \to M$ given by the formula:

$$f_a(x) := F(x, a) \quad \text{for all } x \in M,\ a \in S.$$

We shall show that $\operatorname{card} J = 1$. Since

$$F(y, a_{k+1}) = F\big(\cdots F\big(F(\cdots F(y, a_2)\cdots, a_k), a_{k+1}\big)\cdots, a_{n+1}\big)$$

for all $y \in M$, $a_2, \ldots, a_{n+1} \in S$, we have

$$F\big(F(x, a_1), a_{k+1}\big) = F\big(\cdots F\big(F(\cdots F(F(x, a_1), a_2)\cdots, a_k), a_{k+1}\big)\cdots, a_{n+1}\big)$$

for all $x \in M$, $a_1, \ldots, a_{n+1} \in S$. Moreover

$$F(x, a_k) = F\big(\cdots F\big(F(\cdots F(F(x, a_1), a_2)\cdots, a_k), a_{k+1}\big)\cdots\ a_n\big)$$

for all $x \in M$ $a_1, \ldots, a_n \in S$. Hence

$$F\big(F(x, a_k), a_{n+1}\big) = F\big(\cdots F\big(F(\cdots F(F(x, a_1), a_2)\cdots, a_k), a_{k+1}\big)\cdots, a_{n+1}\big)$$

for all $x \in M$, $a_1, \ldots, a_{n+1} \in S$. Therefore,

$$F\big(F(x, a_1), a_{k+1}\big) = F\big(F(x, a_k), a_{n+1}\big) \quad \text{for all } x \in M,\ a_1, a_k, a_{k+1}, a_{n+1} \in S.$$

Let $a_1 = a_{n+1} = a$ and $a_k = a_{k+1} = b$. Then

$$F\big(F(x, a), b\big) = F\big(F(x, b), a\big) \quad \text{for all } x \in M,\ a, b \in S.$$

Therefore

$$f_b \circ f_a = f_a \circ f_b \quad \text{for all } f_a, f_b \in J. \tag{4.1}$$

From the definition of J it follows that equation (T‴) may be written in the form

$$f_{a_n} \circ f_{a_{n-1}} \circ \cdots \circ f_{a_k} \circ \cdots \circ f_{a_2} \circ f_{a_1} = f_{a_k}. \tag{4.2}$$

Since F satisfies equation (T‴), equation (4.2) is satisfied for all functions $f_{a_1}, \ldots, f_{a_n} \in J$.

From conditions (4.1) and (4.2) it follows that

$$f_{a_k} = f_{a_n} \circ f_{a_{n-1}} \circ \cdots \circ f_{a_k} \circ \cdots \circ f_{a_2} \circ f_{a_1} = f_{a_n} \circ f_{a_{n-1}} \circ \cdots \circ f_{a_1} \circ \cdots \circ f_{a_3} \circ f_{a_2} = f_{a_1}$$

for all $f_{a_1}, \ldots, f_{a_n} \in J$. Therefore $f_a = f_{a_k}$ for all $a \in S$. Hence $\operatorname{card} J = 1$. Let $f\colon M \to M$ be a function such that $J = \{f\}$. Then, obviously, $F(x, a) = f(x)$ for all $(x, a) \in M \times S$ and $f^n(x) = f(x)$ for all $x \in M$.

Conversely, if there exists a function $f\colon M \to M$ such that $F(x, a) = f(x)$ for all $(x, a) \in M \times S$ and $f^n(x) = f(x)$ for all $x \in M$ then, clearly, F is a solution of equation (T‴). This completes our proof.

References

[1] J. Aczél, *Lectures on functional equations and their applications*, Academic Press, New York 1966.

[2] R. Baer, *Linear algebra and projective geometry*, Academic Press, New York 1952.

[3] W. A. Dudek, *Remarks on n-groups*, Demonstratio Math., 13 (1980), 165—181.

[4] Ł. Piechowicz, S. Serafin, *Solution of the translation equation on some structures*, Zeszyty Naukowe UJ, Prace Mat., 21 (1979), 109—114.

WYŻSZA SZKOŁA PEDAGOGICZNA
INSTYTUT MATEMATYKI
A .ZAWADZKIEGO 13/15
42-200 CZĘSTOCHOWA (POLAND)

Received September 1, 1982