UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXVII

1988

Difference quotients for partial differential equations of higher orders

ZBIGNIEW KOWALSKI

§ **1.** In this paper we shall introduce the following difference quotients of higher orders for the discrete function u_{ij}:

(1.1) (The symmetric difference quotient of the order $2n$) =

$$= \frac{1}{h^{2n}} \cdot \sum_{\alpha=-n}^{+n} -(1)^{n+\alpha} \cdot \binom{2n}{n+\alpha} \cdot u_{i+\alpha,j}, \quad (n = 1, 2, 3, \ldots),$$

(1.2) (The foreward difference quotient of the order $2n-1$) =

$$= \frac{1}{h^{2n-1}} \cdot \sum_{\alpha=-n+1}^{n} (-1)^{n+\alpha} \cdot \binom{2n-1}{n+\alpha-1} \cdot u_{i+\alpha,j}, \quad (n = 1, 2, 3, \ldots),$$

(1.3) (The backward difference quotient of the order $2n-1$) =

$$= \frac{1}{h^{2n-1}} \cdot \sum_{\alpha=-n}^{n-1} (-1)^{n+\alpha+1} \cdot \binom{2n-1}{n+\alpha} \cdot u_{i+\alpha,j}, \quad (n = 1, 2, 3, \ldots),$$

(1.4) (The symmetric difference quotient of the order $2n-1$) = $\frac{1}{2}\cdot$[(the foreward difference quotient of the order $2n-1$)+(the backward difference quotient of the order $2n-1$)], $(n = 1, 2, 3, \ldots)$.

The coefficients of these quotients are the binomial coefficients and possess the following property:

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} = 0. \tag{1.5}$$

The difference quotients (1.1) (1.2) (1.3) and (1.4) can be used to establish the convergence of the difference method for the partial differential equations of the higher orders.

For example, let us consider the non-linear partial differential equation of the fourth order

$$\frac{\partial u}{\partial t} = f\left(t, x, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_p}, \frac{\partial^2 u}{\partial x_1^2}, \dots, \frac{\partial^2 u}{\partial x_p^2}, \frac{\partial^3 u}{\partial x_1^3}, \dots, \frac{\partial^3 u}{\partial x_p^3}, \frac{\partial^4 u}{\partial x_1^4}, \dots, \frac{\partial^4 u}{\partial x_p^4}\right), \tag{1.6}$$

without mixed derivatives, the function $f(t, x, u, \overset{1}{q}, \overset{2}{q}, \overset{3}{q}, \overset{4}{q})$, $x = (x_1, \dots, x_p)$, $\overset{i}{q} = (\overset{i}{q}_1, \dots, \overset{i}{q}_p)$ $(i = 1, 2, 3, 4)$, being of the class C^1 in some $(5p+2)$-dimensional paralellepiped. The solution $u(x_1, \dots, x_p)$ is dependent on p variables, therefore we shall apply the concise notation of our preceding papers [1] and [2], cf. Fig. 1 and Fig. 2.

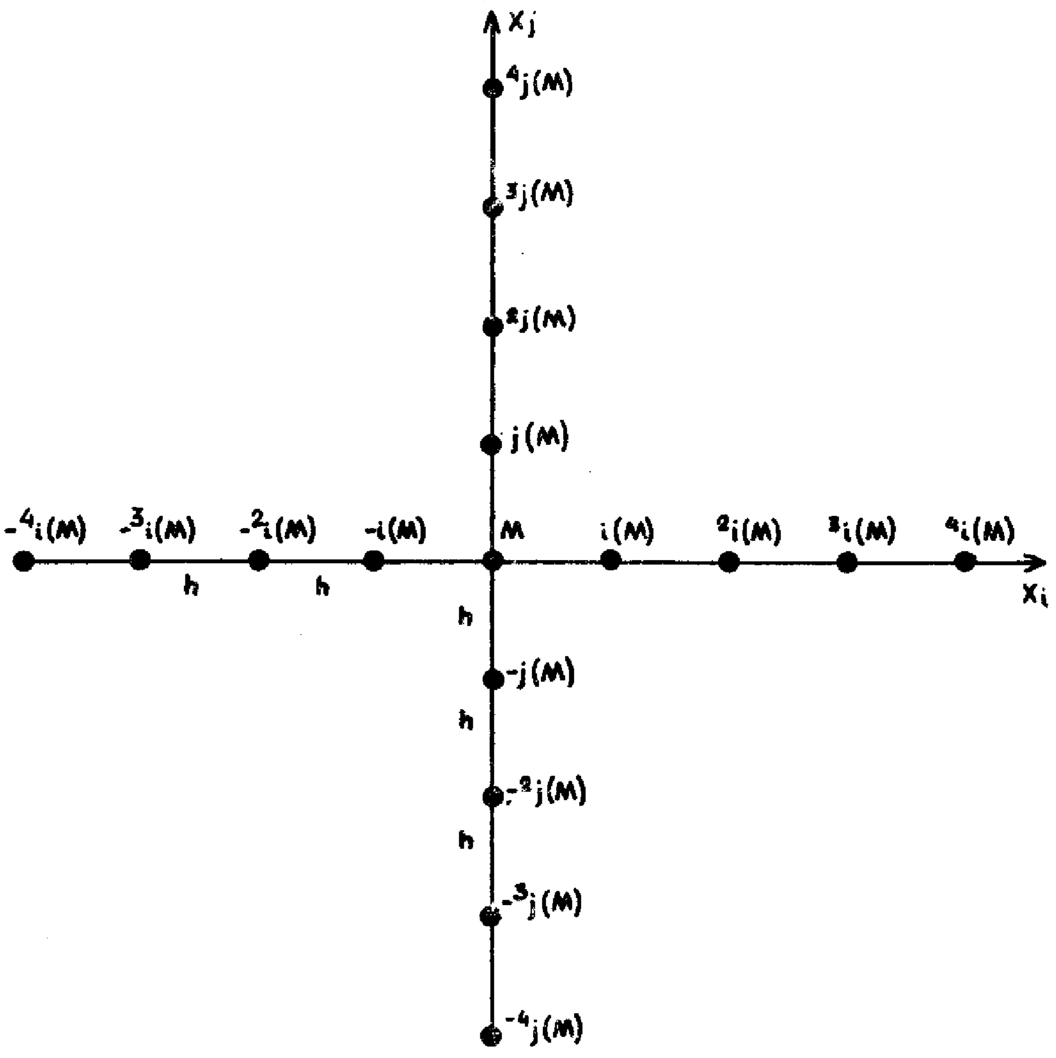

Fig. 1. The nodal points. For the sake of simplicity we have placed the nodal point M at the origin

The solution u^M of the differential equation (1,6) at the nodal points (t^μ, x^m), $M = (\mu, m)$, and the solution v^M of the associated difference equation satisfy as usual the relations

$$u^{M\sim} = f(t^\mu, x^m, u^M, u^{MI}, u^{MII}, u^{MIII}, u^{MIV}) + \eta^M, \tag{1.7}$$

$$v^{M\sim} = f(t^\mu, x^m, v^M, v^{MI}, v^{MII}, v^{MIII}, v^{MIV}). \tag{1.8}$$

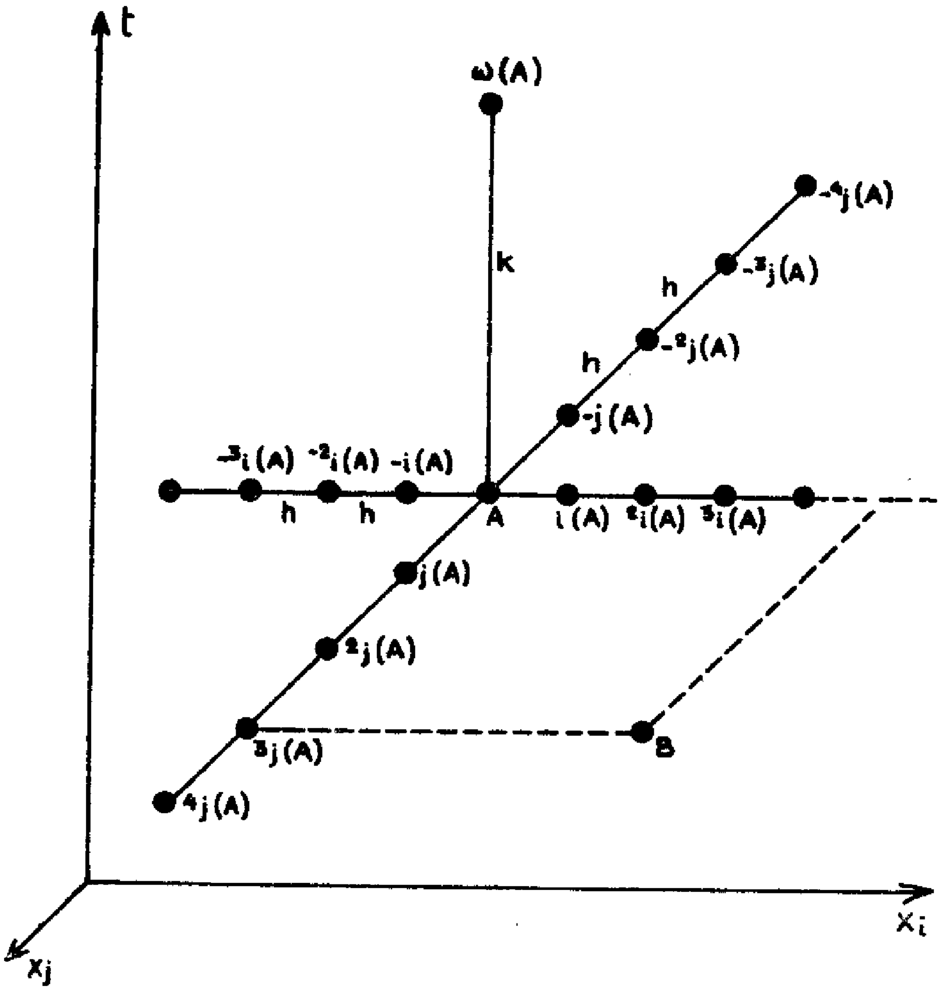

Fig. 2. The nodal points, $\omega(A)$ — max for $r^{\mu+1,m}$, B — max for $r^{\mu,m}$

We introduce the error $r^M = u^M - v^M$, the maximum value $s^\mu = \max\limits_m r^{\mu,m} = r^B$ and we obtain as in the paper [1], § 4, the equality:

$$(1.9)\qquad s^{\mu\sim} = \eta^A + \frac{\partial f}{\partial u}(\sim)\cdot r^A + \sum_{j=1}^{p} \frac{\partial f}{\partial \overset{1}{q}_j}(\sim)\cdot\frac{1}{2h}(r^{j(A)} - r^{-j(A)}) +$$

$$+ \sum_{j=1}^{p} \frac{\partial f}{\partial \overset{2}{q}_j}(\sim)\cdot\frac{1}{h^2}(r^{j(A)} - 2r^A + r^{-j(A)}) +$$

$$+ \sum_{j=1}^{p} \frac{\partial f}{\partial \overset{3}{q}_j}(\sim)\cdot\frac{1}{2h^3}(r^{2j(A)} - 2r^{j(A)} + 2r^{-j(A)} - r^{-2j(A)}) +$$

$$+ \sum_{j=1}^{p} \frac{\partial f}{\partial \overset{4}{q}_j}(\sim)\cdot\frac{1}{h^4}(r^{2j(A)} - 4r^{j(A)} + 6r^A - 4r^{-j(A)} + r^{-2j(A)}) + \frac{1}{k}(r^A - r^B)$$

Then the property (1.5) enables us to introduce in (1.9) the maximum value r^B at appropriate places and we get the equality

$$
\begin{aligned}
(1.10)\qquad s^{\mu\sim} &= \eta^A + \frac{\partial f}{\partial u}(\sim)\cdot r^A + \sum_{j=1}^{p} \frac{\partial f}{\partial \overset{1}{q}_j}(\sim)\cdot\frac{1}{2h}\cdot[(r^{j(A)}-r^B)-(r^{-j(A)}-r^B)]+\\
&+\sum_{j=1}^{p} \frac{\partial f}{\partial \overset{2}{q}_j}(\sim)\cdot\frac{1}{h^2}\cdot[(r^{j(A)}-r^B)-2(r^A-r^B)+(r^{-j(A)}-r^B)]+\\
&+\sum_{j=1}^{p} \frac{\partial f}{\partial \overset{3}{q}_j}(\sim)\cdot\frac{1}{2h^3}[(r^{2j(A)}-r^B)-2(r^{j(A)}-r^B)+2(r^{-j(A)}-r^B)-(r^{-2j(A)}-r^B)]+\\
&+\sum_{j=1}^{p} \frac{\partial f}{\partial \overset{4}{q}_j}(\sim)\cdot\frac{1}{h^4}\cdot[(r^{2j(A)}-r^B)-4(r^{j(A)}-r^B)+6(r^A-r^B)+\\
&-4(r^{-j(A)}-r^B)+(r^{-2j(A)}-r^B)]+\frac{1}{k}\cdot(r^A-r^B)\,.
\end{aligned}
$$

If the derivatives $\partial f/\partial \overset{i}{q}_j$ $(i = 1, 2, 3;\ j = 1, 2, \ldots, p)$ are bounded and the derivatives $\partial f/\partial \overset{4}{q}_j$ $(j = 1, 2, \ldots, p)$ satisfy the assumption

$$
(1.11)\qquad 0 < g \leqslant \frac{\partial f}{\partial \overset{4}{q}_j} \leqslant \mathscr{G} \qquad (j = 1, 2, \ldots, p)\,,
$$

then the terms with h^{-1}, h^{-2} and h^{-3} does not play any significant role in comparison with terms depending on h^{-4}, for sufficiently small h.

Under the assumption (1.11) we have

$$
(1.12)\qquad \sum_{j=1}^{p} \frac{\partial f}{\partial \overset{4}{q}_j}(\sim)\cdot\frac{1}{h^4}\cdot[(r^{2j(A)}-r^B)+6(r^A-r^B)+(r^{-2j(A)}-r^B)] \leqslant 0\,,
$$

hence we can drop these terms in (1.10).

We have also

$$
(1.13)\qquad \sum_{j=1}^{p} \frac{\partial f}{\partial \overset{4}{q}_j}(\sim)\cdot\frac{1}{h^4}\cdot[-4(r^{j(A)}-r^B)-4(r^{-j(A)}-r^B)]+\frac{1}{k}(r^A-r^B) \leqslant 0\,;
$$

the first sum in (1.13) is positive, but we have $r^A - r^B \leqslant 0$, and for a fixed value h we can diminish k so as to obtain the non-positive value for the left-hand side of (1.13). Hence the corresponding terms (1.13) can also be dropped in (1.10) and (1.10) yields the difference inequality

$$
(1.14)\qquad s^{\mu\sim} \leqslant \eta^A + \frac{\partial f}{\partial u}(\sim)\cdot r^A,
$$

and the proof of the convergence of the difference method can be completed as in the papers [1] and [2].

Thus the fact that the difference quotients are defined in an appropriate manner and possess the property (1.5) enables the proof of convergence of the difference method for a partial differential equation of the fourth order.

That fact enables also to complete the proof of convergence for the equation of the order $2n$ $(n = 1, 2, 3, \dots)$ without mixed derivatives. The corresponding calculations will be published in the following paper.

§ 2. The symmetric difference quotients of the order $2n$ $(n = 1, 2, 3, \dots)$

For $n = 1$ we have foreward difference quotient of the first order $\frac{1}{h}(u_{i+1,j} - u_{ij})$, the backward difference quotient of the first order $\frac{1}{h}(u_{ij} - u_{i-1,j})$, and we can define

(2.1) (The symmetric difference quotient of the order 2) =

$$= \frac{1}{h} \cdot \left(\frac{u_{i+1,j} - u_{ij}}{h} - \frac{u_{ij} - u_{i-1,j}}{h} \right) = \frac{1}{h^2} \cdot (u_{i+1,j} - 2u_{ij} + u_{i-1,j}) .$$

In the similar way we can define

(2.2) (The foreward difference quotient of the order 3) =

$$= \frac{1}{h} \cdot \left[\frac{u_{i+2,j} - 2u_{i+1,j} + u_{ij}}{h^2} - \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} \right] =$$

$$= \frac{1}{h^3} (u_{i+2,j} - 3u_{i+1,j} + 3u_{ij} - u_{i-1,j}) ,$$

(2.3) (The backward difference quotient of the order 3) =

$$= \frac{1}{h} \left[\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} - \frac{u_{ij} - 2u_{i-1,j} + u_{i-2,j}}{h^2} \right] =$$

$$= \frac{1}{h^3} \cdot (u_{i+1,j} - 3u_{ij} + 3u_{i-1,j} - u_{i-2,j}) ,$$

and

(2.4) (The symmetric difference quotient of the order 4) = $\frac{1}{h} \cdot$[(the foreward difference quotient of the order 3) – (the backward difference quotient of the order 3)].

Then we obtain

(2.5) (The symmetric difference quotient of the order 4) =

$$=\frac{1}{h^4}\left(u_{i+2,j}-4u_{i+1,j}+6u_{ij}-4u_{i-1,j}+u_{i-2,j}\right).$$

Let us suppose that the difference quotients of the order $2n-1$ are given by the formula (1.2) and (1.3) for a fixed value n.

Then we define

(2.6) (The symmetric difference quotient of the order $2n$) $=\frac{1}{h}\cdot$ [(the foreward difference quotient of the order $2n-1$) $-$(the backward difference quotient of the order $2n-1$)],

and we have

(2.7) (The symmetric difference quotient of the order $2n$) =

$$=\frac{1}{h}\cdot\frac{1}{h^{2n-1}}\cdot\left[u_{i+n,j}+\sum_{\alpha=-n+1}^{n-1}(-1)^{n+\alpha}\cdot\binom{2n-1}{n+\alpha-1}\cdot u_{i+\alpha,j}+\right.$$

$$\left.-\left(-u_{i-n,j}+\sum_{\alpha=-n+1}^{n-1}\cdot(-1)^{n+\alpha+1}\cdot\binom{2n-1}{n+\alpha}\cdot u_{i+\alpha,j}\right)\right].$$

But the coefficient of $u_{i+\alpha,j}$ is of the form

$$(2.8)\qquad (-1)^{n+\alpha}\cdot\binom{2n-1}{n+\alpha-1}-(-1)^{n+\alpha+1}\cdot\binom{2n-1}{n+\alpha}=(-1)^{n+\alpha}\cdot\left[\binom{2n-1}{n+\alpha-1}+\binom{2n-1}{n+\alpha}\right],$$

and

$$(2.9)\qquad \binom{2n-1}{n+\alpha-1}+\binom{2n-1}{n+\alpha}=\binom{2n}{n+\alpha},$$

therefore we have

(2.10) (The symmetric difference quotient of the order $2n$) =

$$=\frac{1}{h^{2n}}\cdot\left[u_{i+n,j}+\sum_{\alpha=-n+1}^{n-1}(-1)^{n+\alpha}\cdot\binom{2n}{n+\alpha}\cdot u_{i+\alpha,j}+u_{i-n,j}\right]=$$

$$=\frac{1}{h^{2n}}\cdot\sum_{\alpha=-n}^{n}(-1)^{n+\alpha}\cdot\binom{2n}{n+\alpha}\cdot u_{i+\alpha,j}.$$

We thus have the result that the formula (1.1) holds.

§ 3. The foreward difference quotients of the order $2n+1$

We shall obtain the difference quotients of the order $2n+1$ with the aid of the symmetric difference quotients of the order $2n$. Let us define

(3.1) (The foreward difference quotient of the order $2n+1$) =

$$=\frac{1}{h}\cdot\left[\frac{1}{h^{2n}}\cdot\sum_{\alpha=-n+1}^{n+1}(-1)^{n+\alpha-1}\cdot\binom{2n}{n+\alpha-1}\cdot u_{i+\alpha,j}+\right.$$

$$\left.-\frac{1}{h^{2n}}\sum_{\alpha=-n}^{n}(-1)^{n+\alpha}\cdot\binom{2n}{n+\alpha}\cdot u_{i+\alpha,j}\right].$$

Then we can write

(3.2) (The foreward difference quotient of the order $2n+1$) =

$$=\frac{1}{h^{2n+1}}\cdot\left[u_{i+n+1,j}+\sum_{\alpha=-n+1}^{n}(-1)^{n+\alpha-1}\cdot\binom{2n}{n+\alpha-1}\cdot u_{i+\alpha,j}+\right.$$

$$\left.-\left(u_{i-n,j}+\sum_{\alpha=-n+1}^{n}(-1)^{n+\alpha}\cdot\binom{2n}{n+\alpha}\cdot u_{i+\alpha,j}\right)\right].$$

The coefficient of $u_{i+\alpha,j}$ can be written in the form

(3.3) $$(-1)^{n+\alpha-1}\cdot\binom{2n}{n+\alpha-1}-(-1)^{n+\alpha}\cdot\binom{2n}{n+\alpha}=$$

$$=(-1)^{n+\alpha-1}\cdot\left[\binom{2n}{n+\alpha-1}+\binom{2n}{n+\alpha}\right]=(-1)^{n+\alpha-1}\cdot\binom{2n+1}{n+\alpha},$$

hence we get

(3.4) (The foreward difference quotient of the order $2n+1$) =

$$=\frac{1}{h^{2n+1}}\cdot\left[u_{i+n+1,j}+\sum_{\alpha=-n+1}^{n}(-1)^{n+\alpha-1}\cdot\binom{2n+1}{n+\alpha}\cdot u_{i+\alpha,j}-u_{i-n,j}\right]=$$

$$=\frac{1}{h^{2n+1}}\cdot\sum_{\alpha=-n}^{n+1}(-1)^{n+\alpha-1}\cdot\binom{2n+1}{n+\alpha}\cdot u_{i+\alpha,j}\,.$$

We have $(-1)^{n+\alpha-1}=(-1)^{n+\alpha+1}$, hence

(3.5) (The foreward difference quotient of the order $2n+1$) =

$$=\frac{1}{h^{2n+1}}\cdot\sum_{\alpha=-n}^{n+1}(-1)^{n+\alpha+1}\cdot\binom{2n+1}{n+\alpha}\cdot u_{i+\alpha,j}\,,$$

cf. the formula (1.2).

§ 4. The backward difference quotient of the order $2n+1$

We shall obtain the backward difference quotient of the order $2n+1$ with the aid of the symmetric difference quotients of the order $2n$.

Let us define

(4.1) (The backward difference quotient of the order $2n+1$) =

$$= \frac{1}{h}\cdot\left[\frac{1}{h^{2n}}\cdot\sum_{\alpha=-n}^{n}(-1)^{n+\alpha}\cdot\binom{2n}{n+\alpha}\cdot u_{i+\alpha,j} + \right.$$

$$\left. -\frac{1}{h^{2n}}\cdot\sum_{\alpha=-n-1}^{n-1}(-1)^{n+\alpha+1}\cdot\binom{2n}{n+\alpha+1}\cdot u_{i+\alpha,j}\right].$$

Then we have

(4.2) (The backward difference quotient of the order $2n+1$) =

$$= \frac{1}{h^{2n+1}}\cdot\left[u_{i+\alpha,j} + \sum_{\alpha=-n}^{n-1}(-1)^{n+\alpha}\cdot\binom{2n}{n+\alpha}\cdot u_{i+\alpha,j} + \right.$$

$$\left. -\left(u_{i-n-1,j} + \sum_{\alpha=-n}^{n-1}(-1)^{n+\alpha+1}\cdot\binom{2n}{n+\alpha+1}\cdot u_{i+\alpha,j}\right]\right..$$

The coefficient of $u_{i+\alpha,j}$ has the form

$$(4.3)\quad (-1)^{n+\alpha}\binom{2n}{n+\alpha}-(-1)^{n+\alpha+1}\cdot\binom{2n}{n+\alpha+1}=(-1)^{n+\alpha}\cdot\left[\binom{2n}{n+\alpha}+\binom{2n}{n+\alpha+1}\right]=$$

$$=(-1)^{n+\alpha}\cdot\binom{2n+1}{n+\alpha+1},$$

so that

(4.4) (The backward difference quotient of the order $2n+1$) =

$$= \frac{1}{h^{2n+1}}\cdot\left[u_{i+n,j} + \sum_{\alpha=-n}^{n-1}(-1)^{n+\alpha}\cdot\binom{2n}{n+\alpha+1}\cdot u_{i+\alpha,j} - u_{i-n-1,j}\right],$$

hence

(4.5) (The backward difference quotient of the order $2n+1$) =

$$= \frac{1}{h^{2n+1}}\cdot\sum_{\alpha=-n-1}^{n}(-1)^{n+\alpha}\cdot\binom{2n+1}{n+\alpha+1}\cdot u_{i+\alpha,j},$$

cf. the formula (1.3).

So starting with (1.2) and (1.3) for some fixed value n we obtain (1.1) for the same value n, because of the definition (2.7), and the formulas (1.2) (1.3) for the next value $n+1$, because of the definitions (3.1) and (4.1).

Thus the formulas (1.2), (1.3) for $n = 1$ and the definitions (2.7), (3.1) and (4.1) yield the difference quotients (1.1), (1.2), (1.3) for all values $n = 1, 2, 3, \ldots$

The mixed difference quotients of the higher orders will be treated in subsequent papers.

References

[1] Z. Kowalski, *A difference method for a non-linear parabolic differential equation without mixed derivatives*, Ann. Polon. Math., 20 (1968), 167—177.

[2] Z. Kowalski, *A difference method for a non-linear system of parabolic differential equations without mixed derivatives*, Bull. Acad. Polon. Sci.: Sér. Sci. Math. Astr. Phys., 15 (1967), 683—689.

Received November 8, 1982