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Differential equations of the fourth order and the convergence of the difference methods

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§ 1. Let us consider the equation

$$\frac{\partial u}{\partial t}=f\left(t, x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^3 u}{\partial x^3}, \frac{\partial^4 u}{\partial x^4}\right), \tag{1.1}$$

where $u = u(t, x)$, $x = (x_1, x_2, \ldots, x_p)$ and

$$\frac{\partial^i u}{\partial x^i}=\left(\frac{\partial^i u}{\partial x_1^i}, \frac{\partial^i u}{\partial x_2^i}, \ldots, \frac{\partial^i u}{\partial x_p^i}\right) \quad (i = 1, 2, 3, 4). \tag{1.2}$$

The derivatives at both sides of (1.1) can be replaced by difference expressions with the space interval h and time interval k as in the paper [1].

But it turns out that the members depending on h^{-1}, h^{-2} and h^{-3} does not play any significant role and the method of the proof can be explained in a short form when the equation

$$\frac{\partial u}{\partial t}=f\left(t, x, u, \frac{\partial^4 u}{\partial x^4}\right), \tag{1.3}$$

is regarded. Therefore let us collect first in § 2 the corresponding assumptions.

§ 2. We shall assume that the function $f(t, x, \overset{4}{q})$, $\overset{4}{q} = (\overset{4}{q}_1, \overset{4}{q}_2, \ldots, \overset{4}{q}_p)$ is of the class C^1 in the set

$$\mathscr{D}_1 = \{0 \leqslant t \leqslant T,\ 0 \leqslant x_j \leqslant a,\ -\infty < u < +\infty,\ -\infty < \overset{4}{q}_j < +\infty \quad (j = 1, \ldots, p)\}.$$

We consider the boundary problem in the set

$$\mathscr{D}\colon\ 0 \leqslant t \leqslant T,\ 0 \leqslant x_j \leqslant a\ (j = 1, 2. \ldots, p)\colon$$

$$\frac{\partial u}{\partial t}=f\left(t, x, u, \frac{\partial^4 u}{\partial x^4}\right), \tag{2.1}$$

$$(2.2)\qquad \begin{cases} u(0,x) = \varphi_0(x), \\ u(t,x) = \varphi_j(t,x), & \text{for } x_j = 0, \\ u(t,x) = \psi_j(t,x), & \text{for } x_j = a, \\ \dfrac{\partial u}{\partial x} = \gamma_j(t,x), & \text{for } x_j = 0, \\ \dfrac{\partial u}{\partial x} = \delta_j(t,x), & \text{for } x_j = a, \\ (j = 1, 2, \dots, p) \end{cases}$$

We assume that the solution $u(t,x)$ of the problem (2.1), (2.2) exists and is of the class C^4 in the set $\mathscr{D}$.

We assume also that

$$(2.3)\qquad 0 < g_4 \leqslant \frac{\partial f}{\partial \overset{4}{q}_j} \leqslant \mathscr{G}_4 \quad \text{in the set } \mathscr{D}_1 \quad (j = 1, \dots, p),$$

$$(2.3\text{a})\qquad 0 \leqslant \frac{\partial f}{\partial u} \leqslant \mathscr{L} \quad \text{in the set } \mathscr{D}_1.$$

The corresponding difference equation of the explicite type will be written in the form

$$(2.4)\qquad \frac{1}{k}\left[v^{\omega(M)} - \frac{1}{p}\sum_{j=1}^{p} \frac{1}{2}\left(v^{j(M)} + v^{-j(M)}\right)\right] = f(t^{\mu}, x^{m}, v^{M}, v^{M\bar{4}}).$$

Here we use the notations of the paper [2], cf. Fig. 1, and $v^{M\bar{4}}$ denotes the vector of the symmetric difference quotients of the fourth order as in the paper [1].

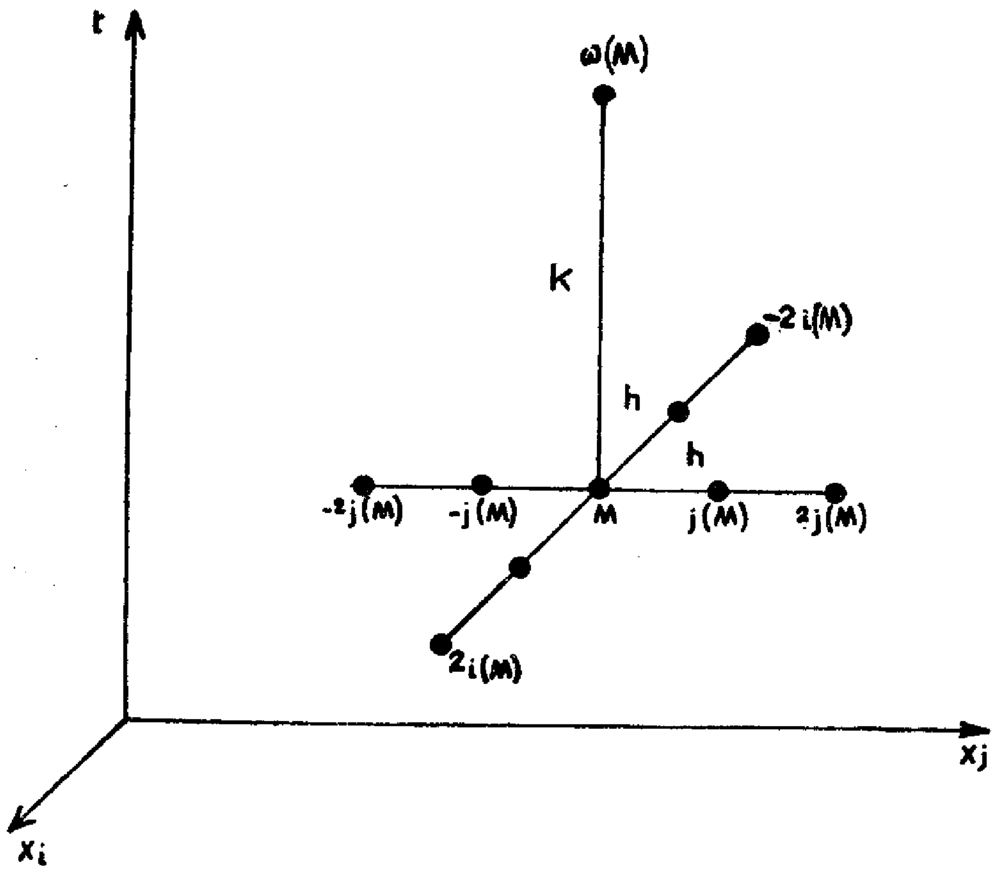

Fig. 1. The nodal points with indices M, $j(M)$, ${}^2j(M)$, $-j(M)$, $-{}^2j(M)$ and $\omega(M)$

The symmetric difference quotients of the first, second and third order will be denoted by

$$\text{(2.4a)}\quad \begin{cases} v^{M\overline{1}} = (v^{M1}, v^{M2}, \dots, v^{Mp}), \\ v^{M\overline{2}} = (v^{M11}, v^{M22}, \dots, v^{Mpp}), \\ v^{M\overline{3}} = (v^{M111}, v^{M222}, \dots, v^{Mppp}), \\ \dots \quad \dots \end{cases}$$

The boundary conditions are induced by (2.2) and have the form

$$\text{(2.5)}\quad \begin{cases} v^{M} = \varphi_0(x^m), & \text{for } M = (0, m), \\ v^{M} = \varphi_j(t^\mu, x^m), & \text{for } m_j = 0, \\ v^{M} = \psi_j(t^\mu, x^m), & \text{for } m_j = N, \\ v^{Mj} = \gamma_j(t^\mu, x^m), & \text{for } m_j = 0, \\ v^{Mj} = \delta_j(t^\mu, x^m), & \text{for } m_j = N, \\ (j = 1, 2, \dots p), & \end{cases}$$

cf. Fig. 2.

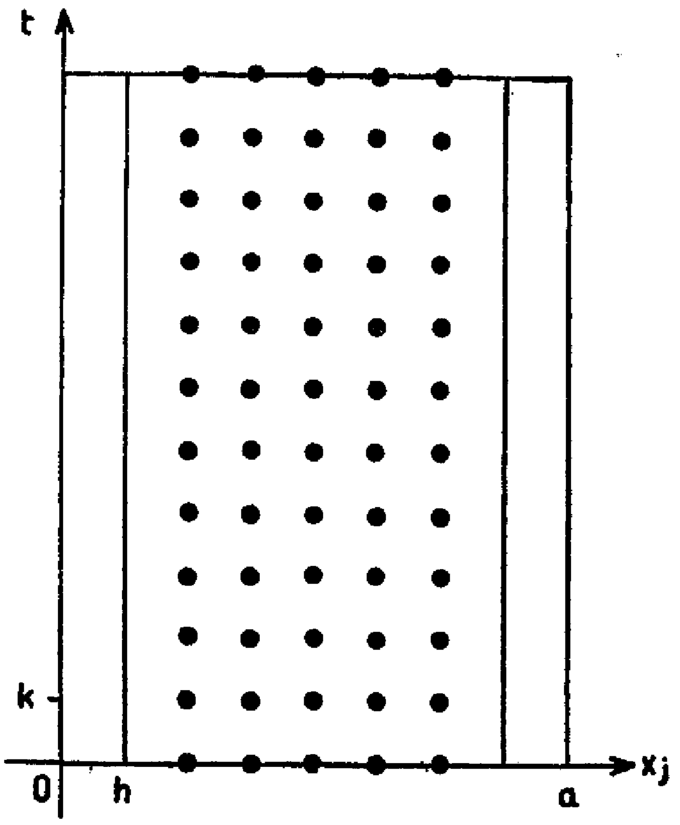

Fig. 2. It is the best way to identify the values u^M and v^M at the nodal points on 2 vertical lines $x_j = 0$ and $x_j = h$. Here the foreward difference quotients of the first order can be taken. There is a similar situation on the lines $x_j = a$ and $x_j = a - h$

The mesh size h for the space coordinates x_j $(j = 1, 2, \dots, p)$ and k for the time coordinate t satisfy the condition

$$\text{(2.6)}\quad \frac{1}{\;}\cdot\frac{1}{\;} - \frac{4}{\;}\cdot\mathscr{G}_4 \geq 0,$$

or

$$k \leqslant \frac{1}{4}\cdot\frac{1}{2p}\cdot\frac{1}{\mathscr{G}_4}\cdot h^4 \tag{2.6a}$$

We define the error η^M by

$$\frac{1}{k}\cdot\left[u^{\omega(M)}-\frac{1}{p}\sum_{j=1}^{p}\frac{1}{2}\left(u^{j(M)}+u^{-j(M)}\right)\right]=f(t^\mu, x^m, u^M, u^{M\bar{4}})+\eta^M, \tag{2.7}$$

and we have

$$\varepsilon(h,k)\to 0, \text{ as } h,k\to 0, \tag{2.8}$$

where

$$\varepsilon(h,k) = \max_M |\eta^M|. \tag{2.9}$$

(2.8) means that the difference equation is consistent with the differential equation.

We define also the error

$$r^M = u^M - v^M. \tag{2.10}$$

§ 3. THEOREM 1. *Under the assumptions of* § 2 *the difference method is convergent.*

Proof. Let us introduce

$$s^\mu = \max_m r^{\mu,m} = r^{\mu,b} = r^B, \tag{3.1}$$

$$s^{\mu+1} = \max_m r^{\mu+1,m} = r^{\mu+1,a} = r^{\omega(A)}. \tag{3.2}$$

We can write

$$s^{\mu\sim} = \frac{1}{k}(s^{\mu+1}-s^\mu) = \frac{1}{k}(r^{\omega(A)}-r^B), \tag{3.3}$$

or

$$s^{\mu\sim} = \frac{1}{k}\cdot\left[r^{\omega(A)}-\frac{1}{p}\sum_{j=1}^{p}\frac{1}{2}(r^{j(A)}+r^{-j(A)})\right]+\frac{1}{k}\cdot\left[\frac{1}{p}\sum_{j=1}^{p}\frac{1}{2}(r^{j(A)}+r^{-j(A)})-r^B\right]. \tag{3.4}$$

We subtract now the equations (2.7) and (2.4), we apply the mean value theorem and we get

$$\begin{aligned} s^{\mu\sim} = {} & \eta^A+\frac{\partial f}{\partial u}(\sim)\cdot r^A+\sum_{j=1}^{p}\frac{\partial f}{\overset{4}{\partial} q_j}(\sim)\cdot\frac{1}{h^4}\cdot[(r^{2j(A)}-r^B)-4(r^{j(A)}-r^B)+ \\ & +6(r^A-r^B)-4(r^{-j(A)}-r^B)+(r^{-2j(A)}-r^B)]+\frac{1}{k}\cdot\left[\frac{1}{p}\sum_{j=1}^{p}\frac{1}{2}(r^{j(A)}+r^{-j(A)})-r^B\right]. \end{aligned} \tag{3.5}$$

cf. Fig. 3.

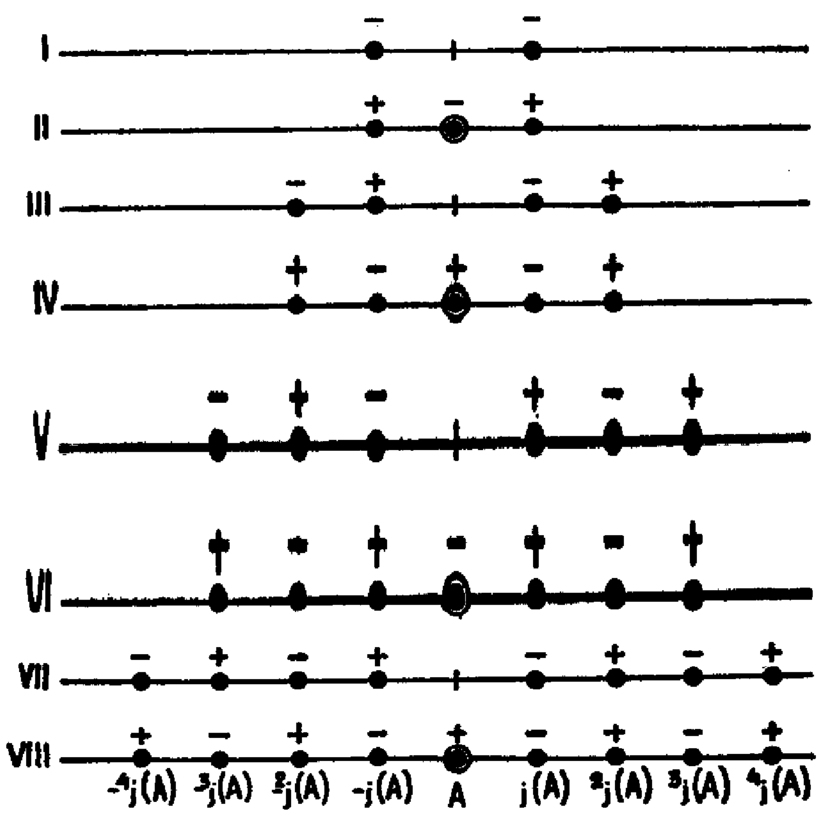

Fig. 3. Symmetric difference quotients of the order I—VIII and the signs of corresponding coefficients. The coefficients in the symmetric difference quotients of the even order can be taken from Pascal's triangle. The coefficients for symmetric difference quotients of the odd order follow by subtraction, cf. [1]

Here we have introduced the maximal value r^B at appropriate places.

But we have

$$r^B = \frac{1}{p} \cdot \sum_{j=1}^{p} r^B, \tag{3.6}$$

hence we can rewrite the last square bracket in (3.5):

$$\frac{1}{p}\sum_{j=1}^{p} \frac{1}{2}(r^{j(A)}+r^{-j(A)})-r^B = \frac{1}{p}\sum_{j=1}^{p}\frac{1}{2}(r^{j(A)}-r^B)+\frac{1}{2}(r^{-j(A)}-r^B). \tag{3.7}$$

We substitute now (3.7) in (3.5), collect terms and obtain

$$\begin{aligned} s^{\mu\sim} &= \eta^A + \frac{\partial f}{\partial u}(\sim)\cdot r^A + \sum_{j=1}^{p} \frac{\partial f}{\overset{4}{\partial} q_j}(\sim)\cdot\frac{1}{h^4}\cdot[(r^{2j(A)}-r^B)+6(r^A-r^B)+ \\ &\quad + (r^{-2j(A)}-r^B)] + \sum_{j=1}^{p}(r^{j(A)}-r^B)\cdot\left[\frac{\partial f}{\overset{4}{\partial} q_j}(\sim)\cdot\frac{-4}{h^4}+\frac{1}{2p}\cdot\frac{1}{k}\right]+ \\ &\quad + \sum_{j=1}^{p}(r^{-j(A)}-r^B)\cdot\left[\frac{\partial f}{\overset{4}{\partial} q_j}(\sim)\cdot\frac{-4}{h^4}+\frac{1}{2p}\cdot\frac{1}{k}\right]. \end{aligned} \tag{3.8}$$

There is no difficulty with terms containing h^{-4} and the nodal points with indices $-{}^{2}j(A)$, A and ${}^{2}j(A)$, since the derivatives $\partial f/\partial \overset{4}{q}_j$ are positive. Hence we have

$$\sum_{j=1}^{p} \frac{\partial f}{\partial \overset{4}{q}_j}(\sim)\cdot\frac{1}{h^4}\left[\left(r^{{}^{2}j(A)}-r^B\right)+6\left(r^A-r^B\right)+\left(r^{-{}^{2}j(A)}-r^B\right)\right]\leqslant 0, \tag{3.9}$$

and we can drop these terms in (3.8).

The terms containing h^{-4} and the nodal points with indices $j(A)$ and $-j(A)$ are more significant because of negative coefficients (-4).

But we have

$$\frac{\partial f}{\partial \overset{4}{q}_j}(\sim)\cdot\frac{-4}{h^4}+\frac{1}{2p}\cdot\frac{1}{k}\geqslant-\frac{4}{h^4}\cdot\mathscr{G}_4+\frac{1}{2p}\cdot\frac{1}{k}\geqslant 0, \tag{3.10}$$

because of the assumptions (2.3) and (2.6). This yields

$$\sum_{j=1}^{p}\left(r^{j(A)}-r^B\right)\cdot\left[\frac{\partial f}{\partial \overset{4}{q}_j}(\sim)\cdot\frac{-4}{h^4}+\frac{1}{2p}\cdot\frac{1}{k}\right]\leqslant 0, \tag{3.11}$$

and

$$\sum_{j=1}^{p}\left(r^{-j(A)}-r^B\right)\cdot\left[\frac{\partial f}{\partial \overset{4}{q}_j}(\sim)\cdot\frac{-4}{h^4}+\frac{1}{2p}\cdot\frac{1}{k}\right]\leqslant 0, \tag{3.12}$$

and these terms can be dropped in (3.8) also.

From (3.8) we get the difference inequality

$$s^{\mu\sim}\leqslant\mathscr{L}\cdot s^{\mu}+\varepsilon(h,k),\ s^0=0, \tag{3.13}$$

and the estimate

$$s^{\mu}\leqslant\frac{\varepsilon(h,k)}{\mathscr{L}}\cdot\left(e^{\mathscr{L}k\mu}-1\right)(\mu=0,1,\ldots,N_1). \tag{3.14}$$

In a similar way we can introduce the minimum values

$$z^{\mu+1}=\min_{m} r^{\mu+1,m}=r^{\mu,c}=r^{\omega(\mathscr{C})}, \tag{3.15}$$

$$z^{\mu}=\min_{m} r^{\eta,m}=r^{\mu,d}=r^{\mathscr{D}}, \tag{3.16}$$

and obtain the inequality

$$z^{\mu}\geqslant-\frac{\varepsilon(h,k)}{\mathscr{L}}\cdot\left(e^{\mathscr{L}k\mu}-1\right)(\mu=0,1,2,\ldots,N_1). \tag{3.17}$$

(3.14) and (3.17) imply that the error estimate

$$|r^M| \leqslant \frac{\varepsilon(h,k)}{\mathscr{L}}\cdot(e^{\mathscr{L}k\mu}-1), \tag{3.18}$$

holds for $\mu = 0, 1, 2, \ldots, N_1$.

The convergence of the method follows from (3.18) and (2.8).

§ **4.** The assumption (2.3a) can be dropped as in the paper [3], cf. [3] Lemma 3 and Lemma 4.

The first, second and third derivatives $\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^3 u}{\partial x^3}$, can be included into the differential equation (1.1) but the proof will be longer to some extent.

The method of proof presented in this paper can be applied with reasonable changes to the equation of the sixth order and for the equation of the order $2n$ ($n = 1, 2, 3, \ldots$) without mixed derivatives. The corresponding calculations will be published.

References

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[2] Z. Kowalski, *A difference method for a non-linear parabolic differential equation without mixed derivatives*, Ann. Polon. Math., 20 (1968), 167—177.

[3] Z. Kowalski, *On the difference method for a non-linear system of parabolic differential equations without mixed derivatives*, Bull. Acad. Polon. Sci.: Sér. Math. Astr. Phys., 16 (1968), 303—310.

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