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Differential equations of the sixth order and the convergence of the difference methods

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§ 1. We shall deal with the equation

$$\frac{\partial u}{\partial t} = f\left(t, x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^3 u}{\partial x^3}, \frac{\partial^4 u}{\partial x^4}, \frac{\partial^5 u}{\partial x^5}, \frac{\partial^6 u}{\partial x^6}\right), \tag{1.1}$$

where $u = u(t, x)$, $x = (x_1, x_2, \ldots, x_p)$ and

$$\frac{\partial^i u}{\partial x^i} = \left(\frac{\partial^i u}{\partial x_1^i}, \frac{\partial^i u}{\partial x_2^i}, \ldots, \frac{\partial^i u}{\partial x_p^i}\right) \quad (i = 1, 2, 3, 4, 5, 6). \tag{1.2}$$

The analysis of the proof of convergence of the difference method for the equation (1,1) shows that it is harder to treat the problem of convergence in the case $2n = 6$ (n – odd) than it is to discuss the same problem in the case $2n = 4$ (n-even), cf. [1]. One explanation for the difficulties encountered in the case $2n = 6$ lies in the fact that the central term in the expression for the symmetric difference quotient of order 6 possesses the negative coefficient $-\binom{6}{3} = -20$ and must be treated with great care. The same attention should be called to the remaining two terms with negative coefficients $-\binom{6}{1}$ in the difference quotient of the sixth order. The members depending on $h^{-1}, h^{-2}, \ldots, h^{-5}$ does not play any significant role. The main idea of the proof of convergence can be explained without lengthy calculations involving terms with $h^{-1}, h^{-2}, \ldots, h^{-5}$, therefore we shall restrict ourselves to the equation

$$\frac{\partial u}{\partial t} = f\left((t, x, u, \frac{\partial^6 u}{\partial x^6}\right). \tag{1.3}$$

The corresponding assumptions are given in the next paragraph.

§ 2. We shall assume that the function $f(t, x, u, \overset{6}{q}), \overset{6}{q} = (\overset{6}{q}_1, \overset{6}{q}_2, \ldots, \overset{6}{q}_p)$ is of the class C^1 in the set $\mathscr{D}_1$: $0 \leqslant t \leqslant T$, $0 \leqslant x_j \leqslant a$, $-\infty < u < +\infty$, $-\infty < \overset{6}{q}_j < +\infty$ $(j = 1, 2, \ldots, p)$.

We consider the following boundary problem in the set $\mathscr{D}$: $0 \leqslant t \leqslant T$, $0 \leqslant x_j \leqslant a$ $(j = 1, 2, \ldots, p)$:

(2.1)
$$\frac{\partial u}{\partial t} = f\left(t, x, u, \frac{\partial^6 u}{\partial x^6}\right),$$

(2.2)
$$\begin{cases} u(0, x) = \varphi_0(x), \\ u(t, x) = \varphi_j(t, x), \quad \text{for } x_j = 0, \\ u(t, x) = \psi_j(t, x), \quad \text{for } x_j = a, \\ \dfrac{\partial^i u}{\partial x^i} = \gamma_{ij}(t, x), \quad \text{for } x_j = 0, \\ \dfrac{\partial^i u}{\partial x^i} = \delta_{ij}(t, x), \quad \text{for } x_j = a, \\ (i = 1, 2)\ (j = 1, 2, \ldots, p). \end{cases}$$

We shall assume that the solution $u(t, x)$ of the problem (2.1), (2.2) exists and is of the class C^6 in the set D.

We shall assume also that

(2.3)
$$0 < g_6 \leqslant \frac{\partial f}{\partial q_j^6} \leqslant \mathscr{G}_6 \text{ in the set } \mathscr{D}_1 \quad (j = 1, \ldots, p),$$

and

(2.3a)
$$0 \leqslant \frac{\partial f}{\partial u} \leqslant \mathscr{L} \text{ in the set } \mathscr{D}_1.$$

The difference equation of the explicite type associated with the differential equation (2.1) will be written in the form

(2.4)
$$\frac{1}{k} \cdot \left[v^{\omega(M)} - \frac{1}{p} \sum_{j=1}^{p} \frac{1}{3} (v^{2j(M)} + v^M + v^{-2j(M)}) \right] = f(t^\mu, x^m, v^M, v^{M\bar{6}}).$$

Here we use the notation of the paper [2], cf. Fig. 1, and $v^{M\bar{6}}$ denotes the vector of the symmetric difference quotients of the sixth order as in the paper [3].

The symmetric difference quotients of the first, second, third and higher orders will be denoted by

(2.4a)
$$\begin{cases} v^{M\bar{1}} = (v^{M1}, v^{M2}, \ldots, v^{Mp}), \\ v^{M\bar{2}} = (v^{M11}, v^{M22}, \ldots, v^{Mpp}), \\ v^{M\bar{3}} = (v^{M111}, v^{M222}, \ldots, v^{Mppp}), \\ \ldots \ldots \end{cases}$$

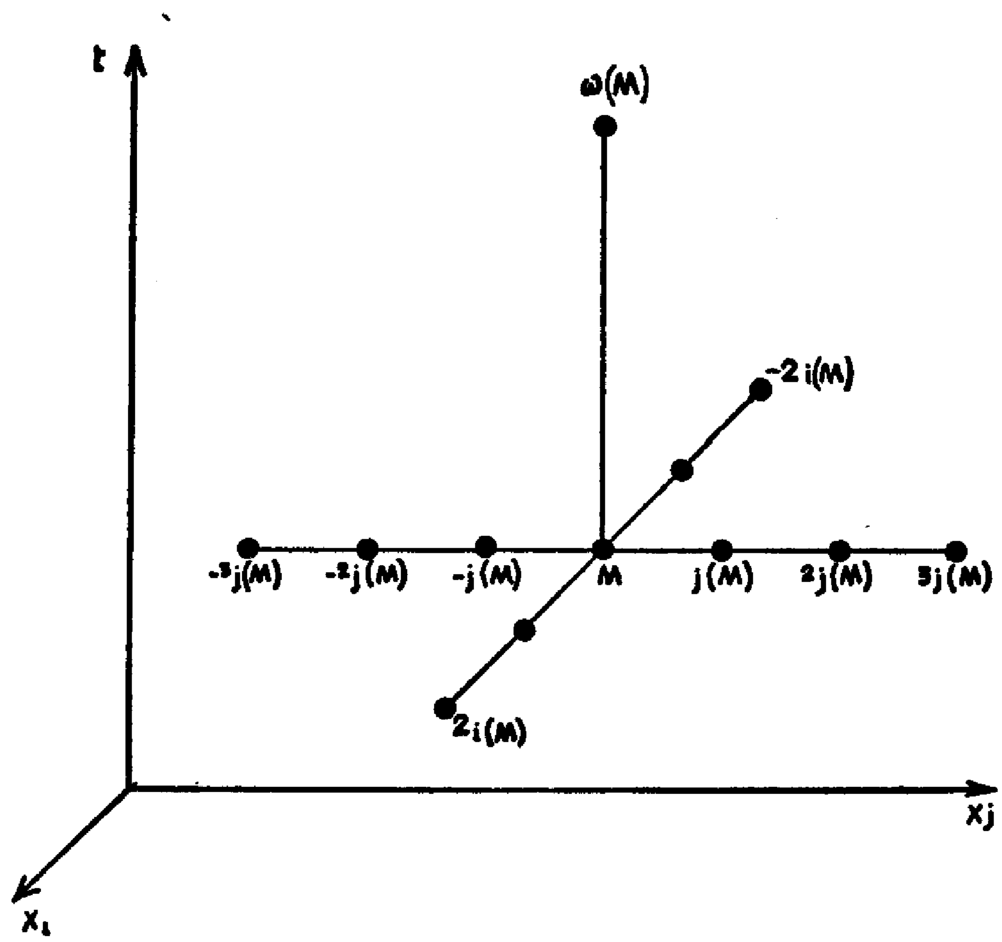

Fig. 1. The nodal points with indices $M, j(M), -j(M), {}^2j(M), -{}^2j(M), -{}^3j(M)$ and $\omega(M)$

The boundary conditions for the difference equation (2.4) are induced by the boundary conditions (2.2) and have the form

$$
(2.5) \qquad \begin{cases} v^M = \varphi_0(x^m)\,, & \text{for } M = (0, m)\,, \\ v^M = \varphi_j(t^\mu, x^m)\,, & \text{for } m_j = 0\,, \\ v^M = \psi_j(t^\mu, x^m)\,, & \text{for } m_j = N\,, \\ v^{Mj} = \gamma_{1j}(t^\mu, x^m)\,, & \text{for } m_j = 0\,, \\ v^{Mj} = \delta_{1j}(t^\mu, x^m)\,, & \text{for } m_j = N\,, \\ v^{Mjj} = \gamma_{2j}(t^\mu, x^m)\,, & \text{for } m_j = 0\,, \\ v^{Mij} = \delta_{2j}(t^\mu, x^m)\,, & \text{for } m_j = N\,, \\ \quad (j = 1, 2, \dots, p)\,, & \end{cases}
$$

cf. Fig. 2.

The mesh size h for the space coordinates x_j $(j = 1, 2, \dots, p)$ and k for the time coordinate t satisfy the conditions

$$
(2.6) \qquad \frac{1}{3p}\cdot\frac{1}{k} - \frac{6}{h^6}\cdot\mathcal{G}_6 \geqslant 0\,, \quad \frac{1}{3p}\cdot\frac{1}{k} - \frac{20}{h^6} \geqslant 0\,,
$$

or

$$
(2.6\text{a}) \qquad k \leqslant \frac{1}{20}\cdot\frac{1}{3p}\cdot\frac{1}{\mathcal{G}_6}\cdot h^6\,.
$$

We define the error η^M by

$$
(2.7) \qquad \frac{1}{k}\left[u^{\omega(M)} - \frac{1}{p}\sum_{j=1}^{p}\frac{1}{3}\left(u^{{}^2j(M)} + u^M + u^{-{}^2j(M)}\right)\right] = f(t^\mu, x^m, u^M, u^{M\overline{6}}) + \eta^M,
$$

and we have

(2.8) $$\varepsilon(h, k) \to 0, \text{ as } h, k \to 0,$$

where

(2.9) $$\varepsilon(h, k) = \max_M |\eta^M|.$$

(2.9) means that the difference equation is consistent with the differential equation.

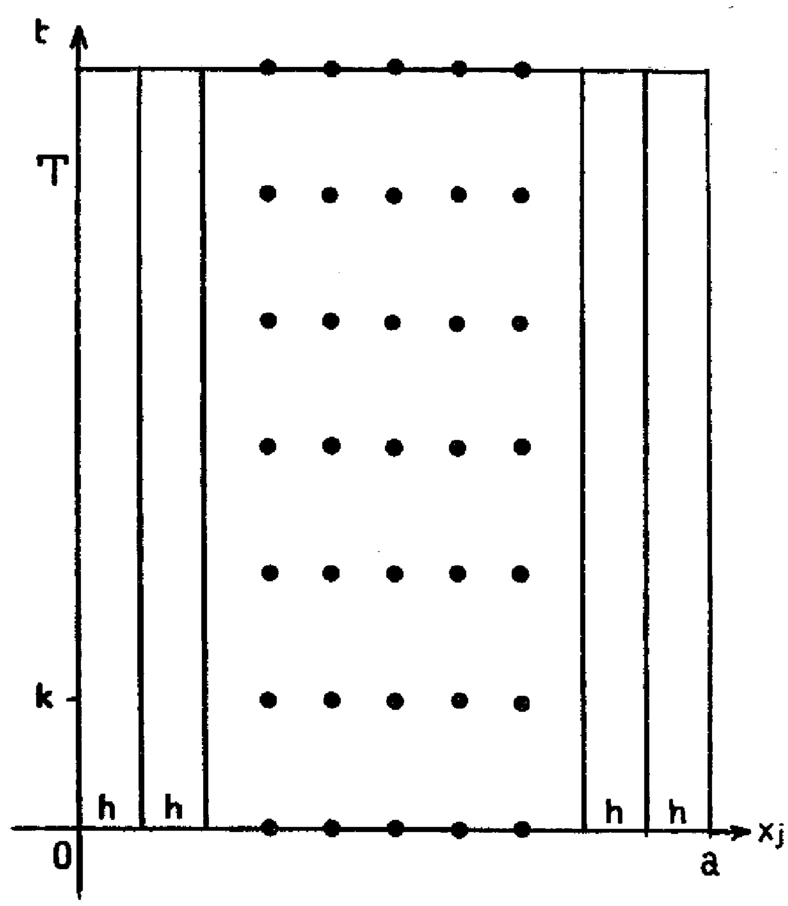

Fig. 2. It is the best way to identify the values u^M and v^M at the nodal points on 3 vertical lines $x_j = 0$, $x_j = h$, $x_j = 2h$. Here the foreward difference quotient of the first order and the symmetric difference quotient of the second order could be taken. There is a similar situation on the lines $x_j = a$, $x_j = a-h$, $x_j = a-2h$

We define also the error

(2.10) $$r^M = u^M - v^M.$$

§ 3. THEOREM 1. *Under the assumptions of* § 2 *the difference method is convergent.*

Proof. We introduce the maximal values

(3.1) $$s^\mu = \max_m r^{\mu, m} = r^{\mu, b} = r^B,$$

(3.2) $$s^{\mu+1} = \max_m r^{\mu+1, m} = r^{\mu+1, a} = r^{\omega(A)}.$$

We can write

(3.3) $$s^{\mu\sim} = \frac{s^{\mu+1} - s^\mu}{k} = \frac{r^{\omega(A)} - r^B}{k},$$

or

$$(3.4)\qquad s^{\mu\sim} = \frac{1}{k}\left[r^{\omega(A)} - \frac{1}{p}\cdot\sum_{j=1}^{p}\frac{1}{3}\left(r^{2j(A)}+r^{A}+r^{-2j(A)}\right)\right]+$$

$$+\frac{1}{k}\left[\frac{1}{p}\sum_{j=1}^{p}\frac{1}{3}\left(r^{2j(A)}+r^{A}+r^{-2j(A)}\right)-r^{B}\right].$$

We substract now the equations (2.7) and (2.4), we apply the mean value theorem and we get

$$(3.5)\qquad \frac{1}{k}\left[r^{\omega(A)}-\frac{1}{p}\sum_{j=1}^{p}\frac{1}{3}\left(r^{2j(A)}+r^{A}+r^{-2j(A)}\right)\right] = \eta^{A}+\frac{\partial f}{\partial u}(\sim)\cdot r^{A}+$$

$$+\sum_{j=1}^{p}\frac{\partial f}{\partial^{6} q_j}(\sim)\cdot\frac{1}{h^{6}}\left[r^{3j(A)}-6r^{2j(A)}+15r^{j(A)}-20r^{A}+15r^{-j(A)}-6r^{-2j(A)}+r^{-3j(A)}\right],$$

cf. Fig. 3.

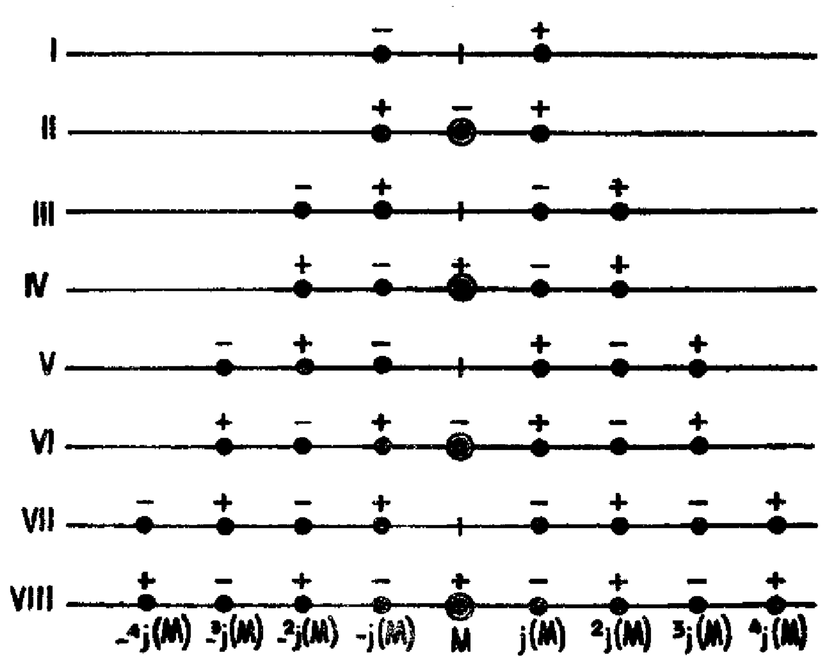

Fig. 3. The coefficients in the symmetric difference quotients of the even order can be taken from Pascal's triangle. The coefficients for symmetric difference quotients of the odd order follow by subtraction, cf. [3]

Introducing the maximal value r^B at appropriate places on the right-hand side of the formula (3.5) we obtain

$$(3.6)\qquad \frac{1}{k}\left[r^{\omega(A)}-\frac{1}{p}\sum_{j=1}^{p}\frac{1}{3}\left(r^{2j(A)}+r^{A}+r^{-2j(A)}\right)\right] = \eta^{A}+\frac{\partial f}{\partial u}(\sim)\cdot r^{A}+$$

$$+\sum_{j=1}^{p}\frac{\partial f}{\partial^{6} q_j}(\sim)\cdot[(r^{3j(A)}-r^{B})-6(r^{2j(A)}-r^{B})+15(r^{j(A)}-r^{B})-$$

$$-20(r^{A}-r^{B})+15(r^{-j(A)}-r^{B})-6(r^{-2j(A)}-r^{B})+(r^{-3j(A)}-r^{B})].$$

We have calculated the first square bracket on the right hand side of the formula (3.4).

The second square bracket in (3.4) can be written as follows. First let us note that we have

$$(3.7) \qquad r^B = \frac{1}{p} \sum_{j=1}^{p} r^B,$$

hence

$$(3.8) \qquad \frac{1}{p} \sum_{j=1}^{p} \frac{1}{3} \left(r^{2j(A)} + r^A + r^{-2j(A)}\right) - r^B =$$

$$= \frac{1}{p} \sum_{j=1}^{p} \frac{1}{3} \left(r^{2j(A)} - r^B\right) + \frac{1}{3}\left(r^A - r^B\right) + \frac{1}{3}\left(r^{-2j(A)} - r^B\right).$$

We can insert now (3.8) and (3.6) into the formula (3.4). This yields

$$(3.9) \qquad s^{\mu\sim} = \eta^A + \frac{\partial f}{\partial u}(\sim)\cdot r^A + \sum_{j=1}^{p} \frac{\partial f}{\partial \overset{6}{q}_j}(\sim)\cdot \frac{1}{h^6}\left[\left(r^{3j(A)} - r^B\right) + 15\left(r^{j(A)} - r^B\right) + \right.$$

$$\left. + 15\left(r^{-j(A)} - r^B\right) + \left(r^{-3j(A)} - r^B\right)\right] + \sum_{j=1}^{p} \left(r^{2j(A)} - r^B\right) \times$$

$$\times \left[\frac{\partial f}{\partial \overset{6}{q}_j}(\sim)\cdot \frac{-6}{h^6} + \frac{1}{3p}\cdot\frac{1}{k}\right] + \sum_{j=1}^{p} \left(r^A - r^B\right) \times$$

$$\times \left[\frac{\partial f}{\partial \overset{6}{q}_j}(\sim)\cdot \frac{-20}{h^6} + \frac{1}{3p}\cdot\frac{1}{k}\right] + \sum_{j=1}^{p} \left(r^{-2j(A)} - r^B\right) \times$$

$$\times \left[\frac{\partial f}{\partial \overset{6}{q}_j}(\sim)\cdot \frac{-6}{h^6} + \frac{1}{3p}\cdot\frac{1}{k}\right].$$

There is no difficulty in the formula (3.9) with terms containing h^{-6} and the nodal points with indices ${}^3j(A)$, $j(A)$, $-j(A)$ and $-{}^3j(A)$ since the derivatives $\partial f/\partial \overset{6}{q}_j$ are positive. Hence we have

$$(3.10) \qquad \sum_{j=1}^{p} \frac{\partial f}{\partial \overset{6}{q}_j}(\sim)\cdot \frac{1}{h^6}\left[\left(r^{3j(A)} - r^B\right) + 15\left(r^{j(A)} - r^B\right) + 15\left(r^{-j(A)} - r^B\right) + \left(r^{-3j(A)} - r^B\right)\right] \leqslant 0,$$

and we can drop these terms in (3.9).

The terms containing h^{-6} and the nodal points with indices ${}^2j(A)$, A, $-{}^2j(A)$ should be treated with greater care.

We have

$$\frac{\partial f}{\partial \overset{6}{q}_j}(\sim)\cdot\frac{-6}{h^6}+\frac{1}{3p}\cdot\frac{1}{k}\geqslant\frac{-6}{h^6}\cdot\mathscr{G}_6+\frac{1}{3p}\cdot\frac{1}{k}\geqslant 0, \tag{3.11}$$

$$\frac{\partial f}{\partial \overset{6}{q}_j}(\sim)\cdot\frac{-20}{h^6}+\frac{1}{3p}\cdot\frac{1}{k}\geqslant\frac{-20}{h^6}\cdot\mathscr{G}_6+\frac{1}{3p}\cdot\frac{1}{k}\geqslant 0, \tag{3.12}$$

because of the assumptions (2.3) and (2.6). This yields

$$\sum_{j=1}^{p}(r^{2j(A)}-r^{B})\cdot\left[\frac{\partial f}{\partial \overset{6}{q}_j}(\sim)\cdot\frac{-6}{h^6}+\frac{1}{3p}\cdot\frac{1}{k}\right]\leqslant 0, \tag{3.13}$$

$$\sum_{j=1}^{p}(r^{A}-r^{B})\cdot\left[\frac{\partial f}{\partial \overset{6}{q}_j}(\sim)\cdot\frac{-20}{h^6}+\frac{1}{3p}\cdot\frac{1}{k}\right]\leqslant 0, \tag{3.14}$$

$$\sum_{j=1}^{p}(r^{-2j(A)}-r^{B})\cdot\left[\frac{\partial f}{\partial \overset{6}{q}_j}(\sim)\cdot\frac{-6}{h^6}+\frac{1}{3p}\cdot\frac{1}{k}\right]\leqslant 0, \tag{3.15}$$

and these terms can be dropped in (3.9) also.

From (3.9) we obtain the difference inequality

$$s^{\mu\sim}\leqslant\mathscr{L}\cdot s^{\mu}+\varepsilon(h,k),\ \varepsilon^0=0, \tag{3.16}$$

and the error estimate

$$s^{\mu}\leqslant\frac{\varepsilon(h,k)}{\mathscr{L}}\cdot(e^{\mathscr{L}k\mu}-1)\ (\mu=0,1,\dots,N_1). \tag{3.17}$$

In a similar way we can introduce the minimum values

$$z^{\mu+1}=\min_m r^{\mu+1,m}=r^{\mu+1,c}=r^{\omega(\mathscr{C})}, \tag{3.18}$$

$$z^{\mu}=\min_m r^{\mu,m}=r^{\mu,d}=r^{\mathscr{D}}, \tag{3.19}$$

and obtain the inequality

$$z^{\mu}\geqslant-\frac{\varepsilon(h,k)}{\mathscr{L}}\cdot(e^{\mathscr{L}k\mu}-1)\ (\mu=0,1,\dots,N_1). \tag{3.20}$$

It follows from the formula (3.17) and (3.20) that the error estimate has the form

$$|r^{M}|\leqslant\frac{\varepsilon(h,k)}{\mathscr{L}}\cdot(e^{\mathscr{L}k\mu}-1)\ (\mu=0,1,\dots,N_1). \tag{3.21}$$

The convergence of the method is the consequence of the formula (2.8) and (3.21).

§ 4. The assumption (2.3a) can be dropped as in the paper [4], cf. [4] Lemma 3 and Lemma 4.

The derivatives $\frac{\partial^i u}{\partial x^i}$ ($i = 1, 2, 3, 4, 5$) can be included into the differential equation (1.1) but the proof will be longer.

The method of proof presented in this paper can be applied to the equation of the order $2n$ ($n = 1, 2, 3, \ldots$) without mixed derivatives. The corresponding calculations will be published.

References

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