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A difference method for the Neumann problem

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§ 1. The purpose of the paper is to investigate the convergence of the difference method for the non-linear problem (3.3), (3.6). The results are based on the method of difference inequalities of the second order, cf. Theorem 1, and are formulated in Theorem 2.

It should be noted that the error estimate (7.2) has the same simple form as in the paper of M. Malec [5], but my paper differs from the paper [5] in so that:

a) the differential equations as considered by M. Malec are special cases of the equation (3.3), since in [5] there is the assumption on the dominating diagonal of the matrix $(f_{w_{ij}})$,

b) the method of proof applied by M. Malec is connected with the regrouping of certain elements of difference quotients so as to obtain the expression with constant sign, which can then be omitted. My method is based on the difference inequalities of the second order.

§ 2. We shall use the notations for the nodal points as in our previous papers [1]—[4]. Let us consider the set

$$Q\colon 0 \leqslant x_j \leqslant \sigma \ (j = 1, \ldots, n),\ 0 < \sigma = \text{const}, \tag{2.1}$$

and the nodal points x^M with coordinates

$$x^M = (x_1^M, x_2^M, \ldots, x_n^M), \tag{2.2}$$

where M denotes the sequence of indices

$$M = (m_1, m_2, \ldots, m_n),\ 0 \leqslant m_j \leqslant N \ (j = 1, \ldots, n) \tag{2.3}$$

and $x_j^M = hm_j \ (j = 1, 2, \ldots, n)$.

The notations for the neighbouring nodal points are the same as in the papers [1]—[4]:

$$x^{i(M)}, x^{ij(M)} \ (i, j = 1, \ldots, n) \tag{2.4}$$

cf. Fig. 1.

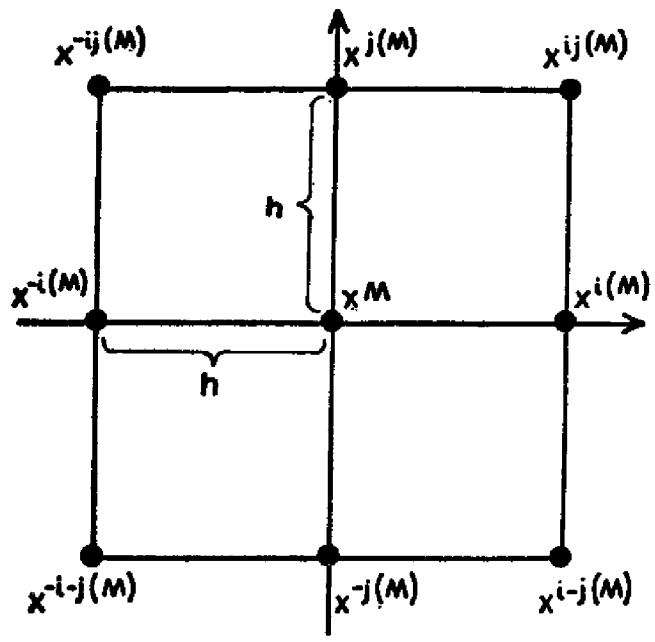

Fig. 1. The nodal points x^M, $x^{i(M)}$, $x^{ij(M)}$... For the sake of simplicity the nodal point x^M has been located at the origin

The difference quotients of the first order will be denoted by

$$u_+^{Mj} = \frac{1}{h}\cdot(u^{j(M)} - u^M)\,, \tag{2.5}$$

$$u_-^{Mj} = \frac{1}{h}(u^M - u^{-j(M)})\,, \tag{2.6}$$

$$u^{Mj} = \frac{1}{2h}(u^{j(M)} - u^{-j(M)})\,, \tag{2.6 bis}$$

and that of the second order by

$$u^{Mjj} = h^{-2}\cdot(u^{j(M)} - 2u^M + u^{-j(M)})\,, \tag{2.7}$$

$$u^{Mij} = \frac{1}{4}h^{-2}\cdot(u^{ij(M)} - u^{-ij(M)} - u^{i-j(M)} + u^{-i-j(M)})\,, \tag{2.8}$$

for $i \neq j$ $(i, j = 1, \ldots, n)$.

We shall use also the difference quotients of the second order:

$$\begin{cases} u_{++}^{Mij} = h^{-2}\cdot(u^{ij(M)} - u^{j(M)} - u^{i(M)} + u^M)\,, \\ u_{-+}^{Mij} = h^{-2}\cdot(u^{j(M)} - u^{-ij(M)} - u^M + u^{-i(M)})\,, \\ u_{--}^{Mij} = h^{-2}\cdot(u^M - u^{-i(M)} - u^{-j(M)} + u^{-i-j(M)})\,, \\ u_{+-}^{Mij} = h^{-2}\cdot(u^{i(M)} - u^M - u^{i-j(M)} + u^{-j(M)})\,, \end{cases} \tag{2.9}$$

as well as the short notation $u^{M\Delta}$ for the vector

$$u^{M\Delta} = (u^{M1}, u^{M2}, \ldots, u^{Mn})\,, \tag{2.10}$$

and the $n \times n$ matrix

$$u^{M\square} = (u^{Mij})\,. \tag{2.11}$$

It will be convenient to replace the coordinates (2.6 bis) of the vector $u^{M\Delta}$ by the difference quotients (2.5) or (2.6).

§ 3. Throughout the paper we shall use the following

ASSUMPTIONS A. 1) We suppose that $f(x, u, q, w)$, $x = (x_1, \dots, x_n) \in R^n$, $u \in R^1$, $q = (q_1, \dots, q_n) \in R^n$, $w = (w_{ij})$, is of the class C^1 in the set (3.1):

$$\begin{cases} 0 \leqslant x_j \leqslant \sigma, \ -\infty < u < +\infty, \ -\infty < q_j < +\infty, \\ -\infty < w_{ij} < +\infty, \ (i, j = 1, \dots, n). \ 0 < \sigma = \text{const}, \end{cases} \tag{3.1}$$

and satisfies the conditions

$$f_{w_{ij}} = f_{w_{ji}}, \tag{3.2}$$

where $f_{w_{ij}} = \partial f / \partial w_{ij}$ $(i, j = 1, \dots, n)$.

2) Let us denote by u_x the vector $u_x = (u_{x_1}, \dots, u_{x_n})$ and by u_{xx} the $n \times n$ matrix $u_{xx} = (u_{x_i x_j})$. We assume that the non-linear differential equation

$$f(x, u, u_x, u_{xx}) = 0, \tag{3.3}$$

is of the elliptic type which means that 1° the quadratic form

$$\sum_{i,j=1}^{n} f_{w_{ij}} \cdot \lambda_i \cdot \lambda_j, \tag{3.4}$$

is positive defined and 2° the derivatives $f_{w_{ij}}, f_{q_j}, f_u$ are bounded:

$$|f_{w_{ij}}| \leqslant \gamma, \ |f_{q_j}| \leqslant \beta, f_u \leqslant \eta < 0, \tag{3.5}$$

in the set (3.1) for $i, j = 1, \dots, n$, γ, β and η being constant.

3) We suppose that there exists the solution $u(x)$ of the equation (3.3) of the class C^2 in the set Q, such that

$$\frac{\partial u}{\partial x_j} = \varphi_j(x)(x_j = 0), \quad \frac{\partial u}{\partial x_j} = \psi_j(x)(x_j = \sigma) \ (j = 1, \dots, n), \tag{3.6}$$

$\varphi_j(x)$ and $\psi_j(x)$ being continuous on the boundary ∂Q, $u(0) = 0$.

In addition we shall suppose that the derivatives $u_{x_i x_j}$ satisfy the Lipschitz condition in the set Q:

$$\begin{cases} |u_{x_i x_j}(x) - u_{x_i x_j}(x')| \leqslant \frac{1}{2}\mathscr{L} \cdot |x_j - x'_j| \\ (i, j = 1, \dots, n), \ x \in Q, \ x' \in Q, \end{cases} \tag{3.7}$$

the points x and x' being on the x_s-axis $(s = 1, \dots, n)$,

$$x = (x_1, \dots, x_n), \ x' = (x'_1, \dots, x'_n), \ x_s \neq x'_s \ (p \neq s, p = 1, \dots, n).$$

4) We suppose that there exists the solution v^M of the difference equation (3.8) associated with (3.3):

$$f(x^M, v^M, v^{M\triangle}, v^{M\square}) = 0, \tag{3.8}$$

and satisfies the conditions

$$v_+^{Mj} = \varphi_j(x^M)(x_j^M = 0), \quad v_-^{Mj} = \psi_j(x^M)(x_j^M = \sigma), v^0 = 0. \tag{3.9}$$

The coordinates of the vector $v^{M\Delta}$ are defined by

$$v^{Mj} = \begin{cases} v_+^{Mj}, & \text{for } b_j^M \geqslant 0, \\ v_-^{Mj}, & \text{for } b_j^M < 0, \end{cases} \tag{3.10}$$

i.e. we use the foreward difference quotient (3.5), if $b_j^M \geqslant 0$, and the backward quotient (3.6), if $b_j^M < 0$, where b_j denotes the derivative f_{q_j}.

We assume in addition that the difference quotients v^{Mij} satisfy the conditions:

$$\begin{cases} |v^{Mjj} - v^{Pjj}| \leqslant \frac{1}{2}h\mathscr{L}, \ |v_{++}^{Mij} - v_{++}^{Pij}| \leqslant \frac{1}{2}h\mathscr{L}, \\ |v_{-+}^{Mij} - v_{-+}^{Pij}| \leqslant \frac{1}{2}h\mathscr{L}, \ |v_{--}^{Mij} - v_{--}^{Pij}| \leqslant \frac{1}{2}h\mathscr{L}, \\ |v_{+-}^{Mij} - v_{+-}^{Pij}| \leqslant \frac{1}{2}h\mathscr{L}, \quad \text{for } h > 0, \end{cases} \tag{3.11}$$

at the nodal points x^M and x^P, $P = s(M)$ ($s = \pm 1, \ldots, \pm n$), the distance between x^M and x^P being h in the direction of the x-axis.

We denote

$$a_{ij} = f_{w_{ij}}, \ b_j = f_{q_j}, \ c = f_u \quad (i, j = 1, \ldots, n). \tag{3.12}$$

§ **4.** The function v^M can be defined at additional nodal points $x_j^M = -h$, $x_j^M = \sigma + h$ ($j = 1, \ldots, n$) as in the paper M. Malec [5]. Then it can be verified that the quantity ε^M:

$$f(x^M, u^M, u^{M\Delta}, u^{M\Box}) = \varepsilon^M, \tag{4.1}$$

possesses the property

$$\varepsilon(h) = \max_{x^M \in Q} |\varepsilon^M| \to 0, \text{ as } h \to 0. \tag{4.2}$$

The simple but lenghtly calculations will be omitted, cf. M. Malec [5].

§ **5.** Lemma 1. *Let us suppose that the Assumptions A are fulfilled and the function $r^M = u^M - v^M$ takes on its maximum value at the nodal point $x^A \in Q$.*

Under these assumptions we have

$$\sum_{i,j=1}^{n} a_{ij}^A \cdot r^{Aij} + \sum_{j=1}^{n} b_j^A \cdot r^{Aj} \leqslant 0, \tag{5.1}$$

u^M and v^M being defined also at additional nodal points, cf. § 4.

Proof. The discrete function r^M takes on its maximum value at the nodal point $x^A \in Q$, therefore the quadratic form

$$\sum_{i,j=1}^{n} r^{Aij} \mu_i \mu_j, \tag{5.2}$$

is negative defined (it is sufficient to introduce the function of the class C^2 which have its maximum value also at x^A, takes on at the nodal points x^M the same values as the function r^M, and possesses at x^A the derivatives of the second order equal to the corresponding difference quotients r^{Aij}).

From the theorem on quadratic forms, cf. M. Krzyżański [6], and from the assumption (3.4) it follows that

$$\sum_{i,j=1}^{n} a_{ij}^{A} \cdot r^{Aij} \leqslant 0 . \tag{5.3}$$

Since in addition the difference quotients r^{Aj} of the first order are taken according to the sign of b_j^A, cf. (3.10), we have $b_j^A \cdot r^{Aj} \leqslant 0$ and

$$\sum_{j=1}^{n} b_j^{A} \cdot r^{Aj} \leqslant 0 . \tag{5.4}$$

From (5.4) and (5.3) it follows the desired result (5.1).

This completes the proof of Lemma 1.

A similar lemma can be proved for the minimum value r^B of the function r^M:

LEMMA 2. *We suppose that the Assumptions A hold. We assume in addition that the function $r^M = u^M - v^M$ takes on its minimum value r^B at the nodal point $x^B \in Q$.*

Under these assumptions we have

$$\sum_{i,j=1}^{n} a_{ij}^{B} \cdot r^{Bij} + \sum_{j=1}^{n} b_j^{B} \cdot r^{Bj} \geqslant 0 . \tag{5.5}$$

The proof of Lemma 2 is similar to the proof of Lemma 1.

§ 6. Now we shall prove the basic Theorem 1 on difference inequalities.

THEOREM 1. *We suppose that the Assumptions A are fulfilled.*

Under these assumptions the discrete function r^M satisfies the following difference inequalities of the second order:

$$\sum_{i,j=1}^{n} a_{ij}^{M} r^{Mij} + \sum_{j=1}^{n} b_j^{M} \cdot r^{Mj} + c^M \cdot r^M \geqslant -\varepsilon(h) , \tag{6.1}$$

$$\sum_{i,j=1}^{n} a_{ij}^{M} \cdot r^{Mij} + \sum_{j=1}^{n} b_j^{M} \cdot r^{Mj} + c^M \cdot r^M \leqslant \varepsilon(h) , \tag{6.2}$$

where a_{ij}^M, b_j^M, c^M denote partial derivatives $f_{w_{ij}}$, f_{q_j}, f_u, respectively, the derivatives being taken at the suitable point $(\sim)$.

Proof. It is sufficient to substract (3.8) from (4.1) and to apply the mean value theorem. We obtain then the equality

$$\sum_{i,j=1}^{n} a_{ij}^{M} \cdot r^{Mij} + \sum_{j=1}^{n} b_j^{M} \cdot r^{Mj} + c^M \cdot r^M = \varepsilon^M , \tag{6.3}$$

which, by the definition (4.2) of $\varepsilon(h)$, yields immediately the desired difference inequalities (6.1) and (6.2).

This completes the proof of Theorem 1.

§ 7. Now we shall prove the Theorem 2 on the convergence of the difference method and the error estimate.

THEOREM 2. *Let us suppose that the Assumptions A are fulfilled, the function* $u(x)$ *is the solution of the Neumann problem* (3.3), (3.6), v^M *is the solution of the associated difference problem* (3.8), (3.9), *and* $r^M = u^M - v^M$.

Under these assumptions 1° *the difference method is convergent*:

$$r^M \to 0, \quad \text{as } h \to 0, \; x^M \in Q, \tag{7.1}$$

2° *we have the error estimate*:

$$|r^M| \leqslant -\frac{\varepsilon(h)}{\eta}, \quad \text{for } x^M \in Q, \tag{7.2}$$

where $\varepsilon(h)$ *is defined by* (4.2).

Proof. From Theorem 1 it follows that the difference inequality (6.1) holds at the nodal point x^A, where r^M achieves its maximum value:

$$\sum_{i,j=1}^{n} a_{ij}^A \cdot r^{Aij} + \sum_{j=1}^{n} b_j^A \cdot r^{Aj} + c^A \cdot r^A \geqslant -\varepsilon(h). \tag{7.3}$$

Let us multiply (5.1) by (-1) and add to (7.3). Then we have

$$c^A \cdot r^A \geqslant -\varepsilon(h). \tag{7.4}$$

The inequality (7.4) can be divided by c^A, since $c^A \leqslant \eta < 0$, cf. (3.11) and (5.5), therefore we have

$$r^A \leqslant -\frac{\varepsilon(h)}{\eta}. \tag{7.5}$$

In the similar way from the difference inequality (6.2) and (5.5) it follows

$$r^B \geqslant +\frac{\varepsilon(h)}{\eta}. \tag{7.6}$$

But we have

$$r^B \leqslant r^M \leqslant r^A, \quad \text{for } x^M \in Q, \tag{7.7}$$

hence the inequality (7.2) follows from (7.5) and (7.6).

The convergence of the method (7.1) is the consequence of (7.2) and the relation (4.2).

This completes the proof of Theorem 2.

§ 8. We might attempt to derive the convergence of the difference method in another way taking into account the fact that there exists the matrix (α_{kl}) $(k, l = 1, \ldots, n)$, such that

$$\sum_{k=1}^{n} \alpha_{ki} \cdot \alpha_{kj} = a_{ij}^A, \quad \sum_{k=1}^{n} \alpha_{kj} = b_j^A. \tag{8.1}$$

With (8.1) in hand we can prove (5.1) without the specific choice of the difference quotients (3.10) of the first order. Then the rest of the paper, i.e. Theorem 1 and Theorem 2, remains unchanged.

But this way does not include all positive defined quadratic forms (3.4), therefore this version will not be presented here.

References

[1] Z. Kowalski, A. Pliś, *Difference inequalities of the elliptic type*, Ann. Polon. Math., 26 (1972), 239—251.

[2] Z. Kowalski, *A difference method for a non-linear elliptic equation with mixed derivatives*, Universitatis Iagellonicae Acta Mathematica, 25 (1985), 215—236.

[3] Z. Kowalski, *Systems of difference inequalities of the elliptic type*, Universitatis Iagellonicae Acta Mathematica, 25 (1985), 237—268.

[4] Z. Kowalski, *A difference method for a non-linear system of elliptic equations with mixed derivatives*, Ann. Polon. Math., 38 (1980), 229—243.

[5] M. Malec, *Sur la méthode des différences finies pour les équations partielles non linéaires avec dérivées mixtes et les conditions aux limites du type de Neumann*, Bull. Acad. Polon. Sci.: Sér. Math. Astr. Phys., 22 (1974), 569—577.

[6] M. Krzyżański, *Partial differential equations*, vols. 1 and 2, Warszawa 1972.

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