

A SIMPLE PROOF OF PUISEUX'S THEOREM

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Abstract. The purpose of this paper is to give a simple topological proof of the following well-known Puiseux Theorem (cf. [2], p.173):

Let A be an irreducible analytic germ at $0 \in \mathbf{C}^n$ of dimension one. Then we can find a system of coordinates z_1, \dots, z_n at $0 \in \mathbf{C}^n$, a representative V of the germ A and a parameterization f for V being a homeomorphism of the form

$$f : D \ni t \longrightarrow (t^p, g(t)) \in V ,$$

where $D := \{t \in \mathbf{C} : |t| < \delta\}$, $\delta > 0$, $p \in \mathbf{N}$ and $g : D \longrightarrow \mathbf{C}^{n-1}$ is a holomorphic mapping with $g(0) = 0$.

The above statement is a generalization of the classical Puiseux Theorem from the middle of the 19th century, which deals with germs of plane algebraic curves in \mathbf{C}^2 (cf. [3], p. 397). The classical version was formulated in terms of "des systèmes circulaires autour d'un point" (cf. [3], p. 388); this term corresponds with the modern notion of a ramified cover.

PROOF OF PUISEUX'S THEOREM. Making a linear change of variables, we can choose a regular system of coordinates z_1, \dots, z_n for the germ A . Thus Rückert's Descriptive Lemma (cf. [2], p. 189) implies that there exists a representative V of the germ A , analytic in $\Omega \times \mathbf{C}^{n-1}$, where $\Omega := \{z_1 \in \mathbf{C} : |z_1| < \varepsilon\}$ ($\varepsilon > 0$), such that

- i) the canonical projection $\psi : V \longrightarrow \Omega$ is proper;
- ii) $\psi^{-1}(0) = \{0\}$;
- iii) the set $V^\circ := V \setminus \{0\}$ is a connected locally topographic manifold over $\Omega^\circ := \Omega \setminus \{0\}$.

Consequently, the restriction $\psi^\circ = \psi|_{V^\circ} : V^\circ \rightarrow \Omega^\circ$ is a finite topological cover with a multiplicity $p \in \mathbf{N}$. Let us consider the following diagram of covers with the multiplicity p

$$\begin{array}{ccc}
 V^\circ & \xleftarrow{\quad f^\circ \quad} & D^\circ \\
 \searrow \psi^\circ & & \swarrow \phi^\circ \\
 & \Omega^\circ &
 \end{array}$$

where $D^\circ := D \setminus \{0\}$, $D := \{t \in \mathbf{C} : |t| < \delta\}$, $\delta := \sqrt[p]{\varepsilon}$ and $\phi^\circ(t) := t^p$.

Obviously, the index of the subgroups $\psi_*^\circ(\pi_1(V^\circ))$ and $\phi_*^\circ(\pi_1(D^\circ))$ in the homotopy group $\pi_1(\Omega^\circ) \simeq \mathbf{Z}$ is p (cf. [1], p.20), whence both these subgroups are equal to $p\mathbf{Z}$. Following the Lifting Criterion (cf. [1], p.22), there exists a homeomorphism $f^\circ : D^\circ \rightarrow V^\circ$ for which the above diagram is commutative (on account of the uniqueness when lifting). Then, of course, $f^\circ(t) = (t^p, g^\circ(t))$ for a mapping $g^\circ : D^\circ \rightarrow \mathbf{C}^{n-1}$.

As the mappings ψ° and ϕ° are local biholomorphisms, so is f° ; hence f° is a biholomorphism. Now, the Riemann Removable Singularity Theorem implies that f° and g° can be extended to the holomorphic mappings $f : D \rightarrow \mathbf{C}^n$ and $g : D \rightarrow \mathbf{C}^{n-1}$ with $f(t) = (t^p, g(t))$.

Since $\psi^{-1}(0) = \{0\}$, we obtain that $f(0) = 0 \in \mathbf{C}^n$ and $g(0) = 0 \in \mathbf{C}^{n-1}$. Therefore we may regard f as a one-to-one mapping onto the set V : $f : D \xrightarrow{\text{one-to-one}} V$; it is proper because so is $\psi \circ f$ (since $(\psi \circ f)(t) = t^p$). This means that f is a closed mapping, and thus – a homeomorphism. The last statement completes our proof.

References

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2. Lojasiewicz S., *Introduction to Complex Analytic Geometry*, Birkhäuser, 1991.
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