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ON THE ASYMPTOTIC BEHAVIOUR OF THE LOGARITHMIC DERIVATIVE OF THE ENTIRE FUNCTION

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The investigations of the boundedness of the composition operators of the special type (see [2]) lead to the elegant condition

$$\limsup_{t\to\infty} \phi'(t)/\phi(t) < +\infty,$$

where ϕ is an analytic function with positive coefficients:

$$(*) \qquad \phi(t) = \sum a_n t^n, \quad a_n \geq 0.$$

This condition would, however, be still more elegant if there were simply $\lim_{t\to\infty} \phi'(t)\phi(t) < +\infty$. This paper shows that, unfortunately, it is impossible, i.e. $\lim_{t\to\infty} \phi'(t)/\phi(t)$ not always exists. Moreover, the following theorem holds:

THEOREM *For any* $0 \leq b < a \leq \infty$ *there exists* ϕ *of the form* (*), *such that*

$$\liminf_{t\to\infty} \phi'(t)/\phi(t) = b,$$
$$\limsup_{t\to\infty} \phi'(t)/\phi(t) = a.$$

PROOF. We may concentrate on the case $b = 0$. Indeed, taking $\widetilde{\phi}(t) = \exp(bt)\phi(t)$ instead of ϕ we can then obtain an arbitrary b.
We will need the functions

$$\psi_{k,a}(t) = 1 + (at/k)^k, \quad a > 0,\ k \in \mathbf{N}.$$

Let $\{a_n\}_{n\in\mathbf{N}} \subseteq \mathbf{R}_+$ be an increasing sequence such that $\lim_{n\to\infty} a_n = a$. Now we are going to recursively define the sequence of natural numbers $\{k_n\}_{n\in\mathbf{N}}$ and the sequence $\{t_n\}_{n\in\mathbf{N}}$ of reals as follows.
Let $k_1 = 1$. Assume that $k_2 \ldots k_n$ and $t_1 \ldots t_{n-1}$ are defined.
Let us denote

$$\psi_n = \psi_{k_n, a_n}, \quad \phi_n = \prod_{j=1}^{n} \psi_j.$$

As ϕ_n is a polynomial, we have $\lim_{t\to\infty} \phi_n'(t)/\phi_n(t) = 0$, so we can find $t_n > t_{n-1} + 1$ such that

$$(1) \qquad \phi'(t)\phi(t) \le 1/n, \quad t \ge t_n.$$

Similarly, since $(c/x)^{x-1} \longrightarrow 0$, $x \to \infty$, we can find k_{n+1} such that

$$(2) \qquad (a_{n+1}t_n/k_{n+1})^{k_{n+1}-1} \le 2^{-n}/a_{n+1}$$

and $k_{n+1} > k_1 + \cdots + k_n$.
Define also $s_n := k_n(k_n - 1)^{1/k_n}/a_n$ and $\beta_n := \psi_n'(s_n)/\psi_n(s_n)$. Notice that

$$(3) \qquad \beta_n = a_n(k_n - 1)^{-1/k_n}(k_n - 1)/k_n = \max\{\psi_n'(t)/\psi_n(t) \colon t \in \mathbf{R}\}$$

and $\lim_{n\to\infty} \beta_n = a$, as k_n tends to infinity.

By Theorem in [1], p. 200, ϕ_n and ϕ_n' tend locally uniformly to ϕ and ϕ', respectively. Moreover, $\phi'/\phi = \sum \psi_j'/\psi_j$.
Using (2) we get

$$\psi_{n+1}'(t)/\psi_{n+1}(t) \le \psi_{n+1}'(t) \le \psi_{n+1}'(t_n) = a_{n+1}(a_{n+1}t_n/k_{n+1})^{k_{n+1}-1} \le 2^{-n}, \qquad t \le t_n$$

and consequently

$$\sum_{j=1}^{\infty} \psi_{n+j}'(t)/\psi_{n+j}(t) \le 2^{-n+1}, \quad t \le t_n,$$

that is

$$(4) \qquad \phi'(t)/\phi(t) \le \phi_n'(t)/\phi_n(t) + 2^{-n+1}, \quad t \le t_n.$$

On the other hand by (1), we get

$$\phi_n'(t)/\phi_n(t) = \phi_{n-1}'(t)/\phi_{n-1}(t) + \psi_n'(t)/\psi_n(t) \le \psi_n'(t)/\psi_n(t) + 1/(n-1), \qquad t \ge t_{n-1},$$

or putting together

$$\phi'(t)/\phi(t) \le \psi_n'(t)/\psi_n(t) + 1/(n-1) + 2^{-n+1},$$
$$t_{n-1} \le t \le t_n.$$

In particular, by (4) and (1), we have

$$\phi'(t_n)/\phi(t_n) \le 1/n + 2^{-n+1}$$

and by (3)

$$\phi'(t)/\phi(t) \le \beta_n + 1/(n-1) + 2^{-n+1},$$
$$t_{n-1} \le t \le t_n.$$

Since t_n and s_n tend to infinity, and β_n tends to a, this shows immediately that

$$\liminf_{t\to\infty} \phi'(t)/\phi(t) = 0,$$
$$\limsup_{t\to\infty} \phi'(t)/\phi(t) \le a.$$

But the last inequality is in fact, an equality, as

$$\phi'(s_n)/\phi(s_n) \ge \psi_n'(s_n)/\psi_n(s_n) = \beta_n.$$

This completes the proof.

References

1. Leja F., *Funkcje zespolone*, Warszawa, 1976.
2. Daniluk A., Stochel J., *Seminormal composition operators induced by affine transformations*, in preparation.

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