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# THE NEWMAN-SHAPIRO ISOMETRY THEOREM – OPERATOR VERSION FOR SYSTEMS

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**Abstract.** Let $S = (S_1, \ldots, S_n)$ be a commuting system of operators in a complex Hilbert space $H$ which fulfills "generalized canonical commutation relation". For an exponential polynomial $\phi$ and an entire vector $h$ of the system $S$ a formula for $\|\phi(S)h\|^2$ is shown. This extends the classical result of Newman-Shapiro from [4].

**1. Introduction.** Let $F_n$ be the Hilbert space of entire functions in $\mathbf{C}^n$ which are square integrable with respect to the Gaussian measure $d\mu = \exp(-|z|^2)dV$, here $dV$ stands for the Lebesgue measure in $\mathbf{C}^n$. The space $F_n$ has the reproducing kernel $e_a(z) = \exp(z, a)$, where $(z, a)$ denotes the scalar product in $\mathbf{C}^n$. The set $\mathcal{P}$ of all complex polynomials in n-variables is dense in $F_n$.

For an entire function $\phi$ the Toeplitz operator $T_\phi$ of multiplication by $\phi$ in $F_n$ has its domain $D(T_\phi)$ given by

$$D(T_\phi) = \{f \in F_n, \quad \phi f \in F_n\}.$$

For a large class of symbols $\phi$, the adjoint operator $T_\phi^*$ can be written as the integral operator $\Pi_{\overline{\phi}}$ defined as

$$\Pi_{\overline{\phi}} f(z) = \int \overline{\phi}(a) f(a) e_a(z) d\mu(a),$$

where $f \in D(T_\phi^*)$ is the domain of $T_\phi^*$ and $\overline{\phi}$ is the complex conjugate of $\phi$.

Let $\|\cdot\|$ be the norm in $F_n$. The Isometry Theorem of Newman-Shapiro [4] is described by the following formula

$$\|\phi f\|^2 = \Sigma_k \|\Pi_{\overline{\phi}^{(k)}} f\|^2 (k^!)^{-1}, \tag{1.1}$$

where $f \in D(T_\phi)$, $\phi^{(k)} = D_1^{k_1} \cdots D_n^{k_n}\phi$, $D_s = \frac{\partial}{\partial z_s}$.
It turns out that (1.1) has an operator analog for a special class of $\phi's$. Namely, for n-commuting operators $S_1, \ldots, S_n$ satisfying the so called "generalized canonical commutation relations" and an exponential polynomial $\phi$ we find below the operator counterpart of formula (1.1).
We found operator version of (1.1) in [3] but only in the case $n = 1$. In this paper we extend this result to arbitrary $n \geq 1$. An application to some boundedness from below problem will also be given.

**2. Isometry Theorem.** The proof of (1.1) given in [4] is based on the Plancherel theorem, and it can not be modified to the operator case. However, formula (1.1) can be proved by a direct algebraic computation if $\phi$ is a polynomial, for n-commuting operators $S_1, \ldots, S_n$ satisfying the so called "generalized canonical commutation relations", see [2]. An algebraic counterpart of (1.1) has already been found in [6].

Below we formulate an extension of (1.1) for the above system $(S_1, \ldots, S_n)$ provided that $\phi$ is an exponential polynomial given by

$$\phi(z) = \Sigma_{s=1}^N p_s(z) \exp(z, a_s) \quad z, a_s \in \mathbf{C}^n, \quad p_s \in \mathcal{P}. \tag{2.1}$$

Before stating the above mentioned extension of (1.1) let us recall the notion of entire vectors.

DEFINITION 2.1. Let $T$ be a linear operator in a complex Hilbert space $H$. We say that $f$ is an entire vector of $T$ if

$$\lim_k \sup[\|T^k f\|(k^!)^{-1}] = 0$$

In what follows we consider a system $S = (S_1, \ldots, S_n)$ of commuting operators which have a dense subspace of joint entire vectors $\mathcal{E}$ (i.e vectors of $\mathcal{E}$ are entire for all $S_i$) and satisfy on $\mathcal{E}$ the "generalized canonical commmutation relations" (GCCR). Namely, we assume that there exist symmetric operators $E_1, \ldots, E_n$ such that

$$[S_i^*, S_j]f = \delta_{ij} E_i^2 f \tag{i}$$