A REMARK ON THE INNER CARATHÉODORY DISTANCE FOR THE ANNULUS

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Abstract. We give a simplified proof of the equality $c_{\mathcal{P}}(x,y) = c^i_{\mathcal{P}}(x,y)$ for $1/R < x, y < R$, where $c_{\mathcal{P}}, c^i_{\mathcal{P}}$ denote the Carathéodory distance and the inner Carathéodory distance for the annulus $\mathcal{P}$, respectively.

For $R > 1$ let $\mathcal{P} = \mathcal{P}(R) = \{z \in \mathbf{C} : \frac{1}{R} < |z| < R\}$ and let $c_{\mathcal{P}}, c^i_{\mathcal{P}}$ denote the *Carathéodory distance* and the *inner Carathéodory distance* for the annulus $\mathcal{P}$, respectively (cf. [Jar-Pfl], pp. 16, 38).
The aim of this note is to present a simplified proof of the following theorem.

THEOREM (cf. [Jar-Pfl], Proposition 5.10). For $\frac{1}{R} < x, y < R$ we have $c_{\mathcal{P}}(x,y) = c^i_{\mathcal{P}}(x,y)$.

PROOF. Let $\frac{1}{R} < a < b < c < R$. It is enough to prove that

$$(*) \qquad c_{\mathcal{P}}(a,b) + c_{\mathcal{P}}(b,c) = c_{\mathcal{P}}(a,c).$$

Let cn, sn, dn denote the functions: *cosinus amplitudinis, sinus ampiltudinis,* and *delta amplitudinis*, respectively (cf. [Whit], p. 291). Put $x = e^{2\pi i t}$ and for $y \in (x, R)$ let $y = e^{2\pi i s}, u := t + s, v := t - s$. One can easily prove (cf. [Jar-Pfl], p. 163) that

$$c_{\mathcal{P}}(x,y) = \tanh^{-1}\left(-ik\frac{\mathrm{cn}(u)\,\mathrm{sn}(v)}{\mathrm{dn}(v)}\right), \quad \text{where } k \text{ is a constant.}$$

Evidently $c_{\mathcal{P}}(x,\cdot) \in \mathcal{C}^\infty((x,R))$. Put

$$G(x,y) := \frac{d}{d\tau}c_{\mathcal{P}}(x,\tau)\big|_{\tau=y}, \quad \frac{1}{R} < x < y < R.$$

Then $G(x,\cdot) \in \mathcal{C}^\infty((x,R))$. Suppose that

$$(**) \qquad G(a,y) = G(b,y), \qquad y \in (b,R).$$

Then

$$\begin{aligned} c_{\mathcal{P}}(a,b) + c_{\mathcal{P}}(b,c) &= \int_a^b G(a,y)\,dy + \int_b^c G(b,y)\,dy \\ &= \int_a^b G(a,y)\,dy + \int_b^c G(a,y)\,dy \\ &= \int_a^c G(a,y)\,dy = c_{\mathcal{P}}(a,c). \end{aligned}$$

Therefore it is enough to prove $(**)$. Put $A :=\mathrm{sn}'(0)$ and $M := \frac{Ak}{2\pi} e^{-2\pi is}$. Then, using classical facts from the theory of amplitudinis functions (cf. [Whit], pp. 292, 295, 296), we get

$$\begin{aligned} G(x,y) = M \Bigg(& \frac{-\mathrm{cn}(u)\,\mathrm{cn}(v) + \mathrm{cn}(u)\,\mathrm{cn}(v)\,\mathrm{dn}^2(v) + \mathrm{cn}(u)\,\mathrm{cn}(v) k^2\,\mathrm{sn}^2(v)}{1 - k^2\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(v)} \\ & + \frac{\mathrm{sn}(u)\,\mathrm{sn}(v)\,\mathrm{dn}(u)\,\mathrm{dn}(v) + \mathrm{cn}(u)\,\mathrm{cn}(v)}{1 - k^2\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(v)} \Bigg) \\ & = M\,\mathrm{cn}(u-v) = \frac{Ak}{2\pi} e^{-2\pi is}\,\mathrm{cn}(2s). \end{aligned}$$

The proof is completed. □

REMARK. The equality $(*)$ was proved in [Jar-Pfl] by complicated algebraic identities (the authors adviced the reader to use a computer). Notice that we have used only basic properties of the amplitudinis functions.

References

[Jar-Pfl] Jarnicki M., Pflug P., *Invariant Distances and Metrics in Complex Analysis*, Walter de Gruyter, Berlin; New York, 1993.

[Whit] Whittaker E.T., Watson G.N., *A Course of Modern Analysis*, vol. 2, PWN, Warsaw, 1968.

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