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TOPOLOGICAL DEGREE METHODS IN FUNCTIONAL BOUNDARY VALUE PROBLEMS

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We present here the existence and multiplicity results for the second order two-point and functional boundary value problems provided the right-hand side of the differential equation satisfies certain sign conditions, and no growth restrictions are required.

Let $\mathbf{X}$ be the Banach space of C^0-functions on $J = [0,1]$ endowed with the sup norm $\|\cdot\|$. Denote by $\mathcal{D}$ the set of all operators $K : \mathbf{X} \to \mathbf{X}$ which are continuous and bounded and by $\mathcal{A}$ the set of all functionals $\gamma : \mathbf{X} \to \Re$ which are linear bounded and increasing. We consider the second order functional differential equation

$$x'' = f(t, x(t), (Fx)(t), x'(t), (Hx')(t)), \quad t \in J, \tag{1}$$

where $f : J \times \Re^4 \to \Re$ satisfies the local Carathéodory conditions and $F, H \in \mathcal{D}$. We work with the Dirichlet, Neumann, and periodic two-point conditions and with the functional conditions of the form

$$\alpha(x) = A, \ x'(1) = B, \qquad A, B \in \Re, \ \alpha \in \mathcal{A}. \tag{2}$$

THEOREM 1. *Suppose that there exist real numbers A, B, L_1, L_2 such that $L_1 \leq B \leq L_2$ and*

$$f(t, x, u, L_1, w) \leq 0 \leq f(t, x, u, L_2, w) \tag{3}$$

for a.e. $t \in J$ and each $(x, u, w) \in [A, B, L_1, L_2, \alpha; F, H]$. Then the problem (1),(2) has a solution u satisfying

$$\|u\| \leq \max\{B - L_1, L_2 - B\} + |A|/\alpha(1) + |B|, L_1 \leq u'(t) \leq L_2 \text{ for } t \in J.$$

We denote

$$[A, B, L, M, \alpha; F, G] = \{(x, u, w) : (x, u, w) \in \Re^3,$$
$$|x| \le \max\{|L - B|, |M - B|\} + |A|/\alpha(1) + |B|,$$
$$|u| \le \sup\{\|Fx\| : \ x(t) \in [0, \max\{|L - B|, |M - B|\} + |A|/\alpha(1) + |B|\},$$
$$|w| \le \sup\{\|Hx\| : x(t) \in [L, M]\}, \text{ for } t \in J\}.$$

We get the similar results for the Dirichlet problem, we need only to add to the condition (3) the second sign condition

$$f(t, x, u, L_4, w) \le 0 \le f(t, x, u, L_3, w), \tag{4}$$

where $L_1 \le B - A \le L_2, L_3 \le B - A \le L_4$.

THEOREM 2. *Let there exist real numbers* r_1, r_2, L_1, L_2 *and* $\mu, \nu \in \{-1, 1\}$ *such that* $r_1 \le r_2, L_1 \le 0 \le L_2$ *and*

$$f(t, r_1, u, 0, w) \le 0 \le f(t, r_2, u, 0, w), \tag{5}$$

$$\nu f(t, x, u, L_1, w) \le 0 \le \mu f(t, x, u, L_2, w) \tag{6}$$

for a.e. $t \in J$ *and each* $(x, u, w) \in [r_1, r_2, L_1, L_2; F, H]$. *Then the periodic or the Neumann problem for the eqaution (1) has a solution* u *with*

$$r_1 \le u(t) \le r_2, \quad L_1 \le u'(t) \le L_2 \quad t \in J.$$

We denote

$$[a, b, L, M; F, H] = \{(x, u, w) : (x, u, w) \in \Re^3,$$
$$|u| \le \sup\{\|Fx\| : x(t) \in [a, b]\},$$
$$|w| \le \sup\{\|Hx\| : x(t) \in [L_1, L_2]\}, \text{ for } t \in J\}.$$

In the case where $f(t, \cdot, u, 0, w)$ changes its sign several times, we can formulate the multiplicity results as well.

References

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