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OSCILLATIONS IN BANACH SPACES

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Let E be a Banach space. For $X, Y \subset E$, we denote by $d(X;Y)$ the distance between the sets X and Y, i.e. the set

$$\inf\{\|x-y\| : x \in X, y \in Y\}.$$

By $\mathbf{T}$ we mean $\mathbf{N}$ or $\mathbf{R}_+$.

DEFINITION 1. Let $T \in \mathbf{T}$, and let $g : \mathbf{T} \times E \to E$ be a semidynamical system. For $Y \subset E$ we define

$$\mathbf{Inv}\,(g,Y) := \{x \in E : g(t,x) \in Y \text{ for } t \in \mathbf{T}\},$$

$$\mathbf{Inv}\,(g,Y,T) := \{x \in E : g(t,x) \in Y \text{ for } t \in \mathbf{T},\ t \leq T\}.$$

DEFINITION 2. Let E be a Banach space and let $C \subset E$ be a cone. Then

$$\mathrm{n}(C) := d(-C \cap \mathrm{S}(0,1); C).$$

If $\mathrm{n}(C) > 0$, then C is called a *normal cone.*

DEFINITION 3. Let E be a Banach space, and let $g : E \times \mathbf{T} \to E$ be a semidynamical system. We assume that z is a stationary point of g. We say that g *oscillates at the vicinity of* z if for every $\varepsilon > 0$ there exist a $T \in \mathbf{T}$ and a neigbourhood U_z of z such that for every normal cone C

$$\mathrm{n}(C) \geq \varepsilon \Rightarrow \mathbf{Inv}\,(g, z+C, T) \cap U_z = \{z\}.$$

PROPOSITION 1. *Let $V_i \subset \mathbf{R}^n$ be normal cones such that $\mathbf{R}^n = \bigcup_{i=0}^{n} V_i$. Assume that there is given a linear dynamical system $g : E \times \mathbf{T} \to \mathbf{R}^n$ which oscillates at the neighbourhood of zero. Then for each $x_0 \in \mathbf{R}^n \setminus \{0\}$ there exists an $n_0 \in \{0, \dots, n\}$ such that*

$$\forall\, t \in \mathbf{T}\ \exists\, t_1, t_2 \geq t : g(t_1, x_0) \in V_{n_0}, g(t_2, x_0) \notin V_{n_0}.$$

We characterize semidynamical systems which oscillate at a given fixed point.

THEOREM 1. *Let E be a Banach space and let $z \in E$. We assume that $f\colon E \to E$ is differentiable at z and that $\mathrm{D}f(z)$ is an invertible operator. Suppose that $f(z) = z$. Then the discrete semidynamical system induced by f oscillates at the vicinity of z if and only if*

$$\sigma(\mathrm{D}f(z)) \cap \mathbf{R}_+ = \emptyset.$$

THEOREM 2. *Let E be a Banach space, and let $z \in E$. We assume that $\Phi : E \to E$ is a Lipschitz vector field such that $\Phi(z) = 0$ and that $\mathrm{D}\Phi(z)$ exists. Then the semidynamical system induced by Φ oscillates at the vicinity of z if and only if*

$$\sigma(\mathrm{D}\Phi(z)) \cap \mathbf{R} = \emptyset.$$

The full version of this paper will appear in the *Journal of the Differential Equations.*

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