

GEOMETRIC DEGREE OF FINITE EXTENSIONS OF PROJECTIONS

BY MAREK KARASÍ

Abstract. Let V, W be algebraic subsets of $\mathbf{k}^n, \mathbf{k}^m$, respectively, with $n \leq m$. It is known (with the only assumption that \mathbf{k} is an infinite field) that any finite polynomial mapping $f : V \rightarrow W$ can be extended to a finite polynomial mapping $F : \mathbf{k}^n \rightarrow \mathbf{k}^m$. The main goal of this paper is to estimate from above the geometric degree of a finite extension $\Pi : \mathbf{k}^k \times \mathbf{k}^n \rightarrow \mathbf{k}^k \times \mathbf{k}^n$ of a projection $\pi : V \rightarrow 0 \times \mathbf{k}^n$, where $V \subset \mathbf{k}^k \times \mathbf{k}^n$. We also present an algorithm to construct the extension Π .

1. Introduction. Let \mathbf{k} be any infinite field, V, W be algebraic subsets of $\mathbf{k}^n, \mathbf{k}^m$, respectively, with $n \leq m$. M. Kwieciński proved (in [Kw]) that any finite mapping $f : V \rightarrow W$ can be extended to a finite mapping $F : \mathbf{k}^n \rightarrow \mathbf{k}^m$. There is a following natural problem: to estimate the number of points in the “generic fiber” of F with the minimal number of points in the “generic fiber”. The author solved (in [MK]) this problem in two cases. The first was for projections from hypersurfaces and the second was for mappings from affine subspaces.

In this paper we consider the projection from an arbitrary irreducible algebraic subset of $\mathbf{k}^k \times \mathbf{k}^n$. The main theorem that we will show says that if $V \subset \mathbf{k}^k \times \mathbf{k}^n$ is an irreducible algebraic set, $\pi : V \rightarrow 0 \times \mathbf{k}^n$ is the natural projection, π is finite and $\pi(V)$ is normal, then there exists a finite mapping $\Pi : \mathbf{k}^k \times \mathbf{k}^n \rightarrow \mathbf{k}^k \times \mathbf{k}^n$ such that $\Pi|_V = \pi$ and $\text{gdeg } \Pi \leq (\text{gdeg } \pi)^k$.

We also present an algorithm for constructing the extension Π . The algorithm is supported by Gröbner basis algorithms. In this paper we also present an example of the mapping f and its extension F with $\text{gdeg } F < \text{gdeg } f$.

2. Preliminary. The coordinate ring of the algebraic set V will be denoted by $\mathbf{k}[V]$. For any polynomial mapping $f = (f_1, \dots, f_n) : V \rightarrow W$ the homomorphism of the coordinate rings $\mathbf{k}[W] \ni P \mapsto P(f_1, \dots, f_n) \in \mathbf{k}[V]$ will be denoted by f^* .

DEFINITION 2.1. A polynomial mapping $f : V \rightarrow W$ is called finite if $\mathbf{k}[V]$ is an integral extension of the ring $f^*(\mathbf{k}[W]) = \mathbf{k}[f_1, \dots, f_m]$.

PROPOSITION 2.2. *If f is finite then there exists a $d \in \mathbf{N}$ such that $\#f^{-1}(y) \leq d$ for every $y \in W$.*

PROOF. The mapping f is finite, so there exist polynomials $P_i = T^{k_i} + a_{1,i}T^{k_i-1} + \dots + a_{k_i,i} \in \mathbf{k}[f_1, \dots, f_m][T]$, $i = 1, \dots, n$, such that

$$x_i^{k_i} + a_{1,i}x_i^{k_i-1} + \dots + a_{k_i,i} = 0$$

in $\mathbf{k}[V]$. For any $x \in V$ we have

$$x_i^{k_i} + a_{1,i}(f_1(x), \dots, f_m(x))x_i^{k_i-1} + \dots + a_{k_i,i}(f_1(x), \dots, f_m(x)) = 0.$$

If $x \in V$, $y \in W$ are such that $f(x) = y$ then

$$x_i^{k_i} + a_{1,i}(y)x_i^{k_i-1} + \dots + a_{k_i,i}(y) = 0 \quad i = 1, \dots, n$$

and so

$$\#\{(x_1, \dots, x_n) \in V \mid f(x_1, \dots, x_n) = y\} \leq k_1 \cdot \dots \cdot k_n = \deg P_1 \cdot \dots \cdot \deg P_n,$$

for $y \in W$. □

REMARK 2.3. Let $f : V \mapsto W$ be finite. If Z is an algebraic subset of V then $f|_Z : Z \mapsto W$ is finite.

PROOF. For any $x \in \mathbf{k}[Z]$ there exists an $\tilde{x} \in \mathbf{k}[V]$ such that $\tilde{x}|_Z = x$. Since f is finite, there exist polynomials a_i such that

$$\tilde{x}^k + a_1(f_1, \dots, f_m)\tilde{x}^{k-1} + \dots + a_k(f_1, \dots, f_m) = 0.$$

By the above

$$x^k + a_1(f_1|_Z, \dots, f_m|_Z)x^{k-1} + \dots + a_k(f_1|_Z, \dots, f_m|_Z) = 0$$

and so x is integral over $\mathbf{k}[f_1|_Z, \dots, f_m|_Z] = (f|_Z)^*(\mathbf{k}[W])$. □

DEFINITION 2.4. A polynomial mapping $f : V \rightarrow W$ is called dominating if $f(V)$ is dense in W (in the Zariski topology).

If $f : V \mapsto W$ is dominating then $f^* : \mathbf{k}[W] \rightarrow \mathbf{k}[V]$ is a monomorphism. In this case we will identify $\mathbf{k}[W]$ with $f^*(\mathbf{k}[W]) \subset \mathbf{k}[V]$. If V and W are irreducible then $\mathbf{k}[W]$ and $\mathbf{k}[V]$ are integral domain and then f^* can be extended to the monomorphism $f^* : \mathbf{k}(W) \mapsto \mathbf{k}(V)$ of fields. Similarly as before we will identify $\mathbf{k}(W)$ with $f^*(\mathbf{k}(W)) \subset \mathbf{k}(V)$.