ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO

Nr 55 PRACE MATEMATYCZNE. ZESZYT 7 1962

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On the convergence to zero of oscillating solutions of an ordinary differential equation of order n

In this paper we are to deal with solutions $y(x)$ of n-order differential equation which tend to zero as $x \to \infty$. In the theory of stability a number of sufficient conditions for the existence of such solutions is known [3].

In the case of second order differential equation

$$y'' = f(x, y, y'), \tag{1}$$

under rather general assumptions it may be proved that the solution $y(x)$ tends to zero, if it oscillates and the oscillations are more and more frequent i.e. if the following conditions hold:

$$y(x_\nu) = 0 \quad \text{for} \quad \nu = 1, 2, \ldots \tag{2}$$

$$x_1 < x_2 < x_3 < \ldots \tag{3}$$

$$\lim_{\nu\to\infty} x_\nu = +\infty \tag{4}$$

$$\lim_{\nu\to\infty} (x_{\nu+1} - x_\nu) = 0\,. \tag{5}$$

With aid of differential inequalities the analogous theorem can be stated for the equation

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), \tag{6}$$

where the function f satisfies the condition

$$|f(x, z_0, z_1, \ldots, z_{n-1})| \leqslant K\left(1 + \sum_{i=0}^{n-1} |z_i|\right) \tag{7}$$

and K is a constant [2].

We give some criterion for the convergence to zero of an oscillating solution of equation (6) replacing the inequality (7) by another estimate. An analogous result for the equation (1) is due to A. Lasota [2].

Theorem. *Assume that the function $f(x, y, z_1, \ldots, z_{n-1})$ is defined in the domain*

$$D_a\colon\ a < x;\ y, z_1, \ldots, z_{n-1}\ \text{arbitrary}$$

and

$$(8) \qquad |f(x, y, z_1, \ldots, z_{n-1})| \leqslant M \quad \text{for} \quad |z_{n-1}| \leqslant h,$$

where M and h are positive constants. Let $y(x)$ be the solution of the equation (6) *satisfying the conditions* (2)-(5) $(x_1 > a)$. *Then*

$$\lim_{x\to\infty} y(x) = 0.$$

Proof. We shall first prove that there is N such that for $\nu \geqslant N$ we have

$$(9) \qquad |y^{(n-1)}(x)| \leqslant h$$

in the interval

$$\Delta_\nu = [x_{\nu+1}, x_{\nu+n}].$$

In fact, suppose that the inequality (9) does not hold for an infinity of intervals Δ_ν. Then in each of these intervals there exists a point at which $|y^{(n-1)}(x)| > h$ and a point at which $y^{(n-1)}(x) = 0$ (this follows from the Rolle's theorem applied to the function $y(x)$ and its derivatives). This implies the existence of an infinity of intervals $[\alpha_\nu, \beta_\nu] \subset \Delta_\nu$ such that

$$(10) \qquad |y^{(n-1)}(x)| < h \quad \text{for} \quad \alpha_\nu < x < \beta_\nu$$

and

$$(11) \quad |y^{(n-1)}(\alpha_\nu)| = h, \quad y^{(n-1)}(\beta_\nu) = 0 \quad \text{or} \quad y^{(n-1)}(\alpha_\nu) = 0, \quad |y^{(n-1)}(\beta_\nu)| = h.$$

Due to (11) and by mean value theorem applied to the function $y^{(n-1)}(x)$ in the interval $[\alpha_\nu, \beta_\nu]$ we get:

$$h = |y^{(n-1)}(\beta_\nu) - y^{(n-1)}(\alpha_\nu)| \leqslant (\beta_\nu - \alpha_\nu) \max_{[\alpha_\nu, \beta_\nu]} |y^{(n)}(x)|.$$

The inequality (10) and the assumption (8) imply

$$\max_{[\alpha_\nu, \beta_\nu]} |y^{(n)}(x)| \leqslant \max_{[\alpha_\nu, \beta_\nu]} |f(x, y, \ldots, y^{(n-1)})| \leqslant M$$

therefore we get the estimate for the length of the intervals

$$(12) \qquad x_{\nu+n} - x_{\nu+1} \geqslant \beta_\nu - \alpha_\nu \geqslant \frac{h}{M} > 0$$

valid for every Δ_ν in which (9) does not hold. This contradicts the assumption (5), because

$$(13) \quad \lim_{\nu\to\infty}(x_{\nu+n} - x_{\nu+1}) = \lim_{\nu\to\infty}(x_{\nu+n} - x_{\nu+n-1}) + \ldots + \lim_{\nu\to\infty}(x_{\nu+2} - x_{\nu+1}) = 0.$$

This means that for sufficiently large ν in the interval Δ_ν the inequality (9) holds.

Now we shall make use of the following lemma [1]:

If a function $\lambda(x)$ is defined and n times differentiable in an interval (a, b) and has n zeros in this interval, that is

$$\lambda(x_k) = 0 \quad \textit{for} \quad k = 1, 2 \dots n,$$

where $a < x_1 < \dots < x_n < b$, then for every $x \in (a, b)$ there exists ξ such that

$$\lambda(x) = (x - x_1)(x - x_2) \dots (x - x_n) \frac{\lambda^{(n)}(\xi)}{n!}$$

and

$$\min(x, x_1) < \xi < \max(x, x_n) \text{ }^{1)}.$$

We turn to the proof of our theorem. Choosing $c = c_\nu$ such that

(14) $$|y(c)| = \max_{\Delta_\nu} |y(x)| \qquad x_{\nu+1} \leqslant c \leqslant x_{\nu+n}$$

and applying the lemma to the function $y(x)$ for $x = c$ we have

(15) $$y(c) = (c - x_{\nu+1})(c - x_{\nu+2}) \dots (c - x_{\nu+n-1}) \frac{y_{(n-1)}(\xi)}{(n-1)!}$$

where

$$x_{\nu+1} < \xi < x_{\nu+n}.$$

From (9), (14) and (15) we get the estimate:

$$\max_{\Delta_\nu} |y(x)| \leqslant (x_{\nu+n} - x_{\nu+1})^{n-1} \frac{h}{(n-1)!}$$

which owing to (13) implies

$$\lim_{\nu \to \infty} \max_{\Delta_\nu} |y(x)| = 0.$$

Hence we have

$$\sup_{x \geqslant x_N} |y(x)| = \sup_{\nu \geqslant N-1} \max_{\Delta_\nu} |y(x)| \to 0 \quad \text{as} \quad N \to \infty$$

thus

$$\lim_{x \to \infty} y(x) = 0$$

which finishes the proof.

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[1]) This lemma may be easily proved applying the Rolle's theorem to the function $\Phi(t) = (t - x_1) \dots (t - x_n) \lambda(x) - (x - x_1) \dots (x - x_n) \lambda(t)$ and its derivatives.

Remark. If (8) does not hold then there may exist a solution satysfying (2)-(5) which does not tend to zero. An example has been given by A. Lasota [2].

REFERENCES

[1] В. Л. Гончаров, *Теория интерполирования и приближения функций*, Москва 1954.

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[3] И. Г. Малкин, *Теория устойчивости движения*, Москва 1952.