ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
Nr 55 PRACE MATEMATYCZNE. ZESZYT 7 1962

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On a generalized capacity of a set in the space of two complex variables

Let R_2 be the space of two complex variables. The coordinates of a point $p \in R_2$ are denoted by z, w. The function

$$h(p, q) = \tfrac{1}{2}(z_1 w_2 - w_1 z_2)$$

of two points $p(z_1, w_1)$, $q(z_2, w_2)$ satisfies the conditions $|h(p, q)| = |h(q, p)|$, $h(p, p) = 0$. It is known [1] that for every bounded closed set $E \in R_2$ there exists the capacity $v(E)$ of the set E. The capacity $v(E)$ can be generalized as follows. Let $h(p, q)$ be a function defined in a domain $G \subset R_2$. Suppose $h(p, q)$ satisfies the conditions

1° $h(p, q)$ is analytic with respect to $p \in G$ and $q \in G$
2° $h(p, q)$ is continuous with respect to the both variables points $p, q \in G$
3° $|h(p, q)| = |h(q, p)|$, $|h(p, p)| = 0$.

Let E be a continuum contained in G and let $p^{(n)} = \{p_0, \dots, p_n\}$ be a system of $n+1$ points on E. Put $\Delta^{(j)}(p^{(n)}) = \prod_{\substack{k=0 \\ k\neq j}}^{n} |h(p_j, p_k)|$ and

$$\Delta_n(E) = \sup_{p^{(n)} \in E} \{\min_j \Delta^{(j)}(p^{(n)})\}.$$

It is known that the limit $\lim_{n\to\infty} \sqrt[n]{\Delta_n(E)} = v_h(E)$ exists. $v_h(E)$ is called the generalized capacity of E with respect to the function h.

Let E be a rectifiable curve $z = z(t)$, $w = w(t)$, $t \in [\alpha, \beta]$ and let $h(p, q)$ be a function which satisfies the conditions 1°-3°.

Theorem. *If $h(p, q_0) \neq 0$ for $p \in E$, $p \neq q_0$ where q_0 is an arbitrary point fixed on E, then $v_h(E) > 0$.*

Proof. Let $p^{(n)} = (p_0, \dots, p_n)$ be a system of $n+1$ points on E such that the distance $|p_j, p_{j+1}|$ of the points p_j, p_{j+1} is equal to L/n where L is the length of the curve E. From the definition of $\Delta_n(E)$ follows $\Delta_n(E) \geqslant \min_j \prod_{k\neq j} |h(p_j, p_k)|$. Through every point $q \in E$ passes a hypersurface $h(p, q) = 0$ because $h(q, q) = 0$. Consider the point p_j for which the product $\Delta^{(j)}(p^{(n)})$ is the smallest one.

For the fixed point p_j and the function $h(p_j, p) = h[(z_j, w_j), (z, w)]$ is $h(p_j, p_j) = 0$. The function $h(p_j, p)$ is analytic with respect to the point p; therefore from the Weierstrass theorem follows that there exists a neighbourhood $V\{|z-z_j| < 2r, |w-w_j| < 2s\}$ such that

$$h[(z_j, w_j), (z, w)] = \Omega(z-z_j, w-w_j)(z-z_j)^l[(w-w_j)^k + \\ + A_1(z-z_j)(w-w_j)^{k-1} + \ldots + A_k(z-z_j)]$$

where $A_k(z-z_j)$ are analytic functions of $z-z_j$ in the circle $|z-z_j| < r$, $A_k(0) = 0$ and $\Omega \neq 0$ in V. Therefore $|\Omega| \geqslant M$, $M > 0$ for $|z-z_j| < r$, $|w-w_j| < s$. Let $(w-w_j)_l = f_l(z-z_j)$, $l = 1, \ldots, k$, be the roots of the polynomial

$$(w-w_j)^k + A_1(z-z_j)\cdot(w-w_j)^{k-1} + \ldots + A_k(z-z_j)\,.$$

The functions $f_l(z-z_j)$, $j = 1, \ldots, k$, are analytic for $|z-z_j| < r$. The function $h(p_j, p)$ can be written in the form

$$h(p_j, p) = \Omega(z-z_j, w-w_j)(z-z_j)^l[w-w_j-f_1(z-z_j)] \ldots [w-w_j-f_k(z-z_j)]\,.$$

As $f_s(z-z_j) = 0$ for $z = z_j$, $s = 1, \ldots, k$, then

$$|w-w_j-f_s(z-z_j)| \geqslant \tfrac{1}{2}|w-w_j| \quad \text{for} \quad |z-z_j| < \delta < r\,, \quad s = 1, 2, \ldots, k\,.$$

Hence

$$|h(p, p_j)| \geqslant M\cdot|z-z_j|^l\cdot(\tfrac{1}{2})^k\cdot|w-w_j|^k, \quad |z-z_j| < \delta\,.$$

If (z, w) does not lie on the plains $z = z_j$, $w = w_j$, then $|z-z_j| > 0$, where $|z-z_j|$ is the distance of the point (z, w) from the plain $z = z_j$. Similar $|w-w_j| > 0$. Therefore $\sqrt{|z-z_j|^2 + |w-w_j|^2} > 0$. Suppose that

$$\frac{|z-z_j|}{\sqrt{|z-z_j|^2 + |w-w_j|^2}} \geqslant a\,, \qquad \frac{|w-w_j|}{\sqrt{|z-z_j|^2 + |w-w_j|^2}} \geqslant \beta\,, \qquad a > 0\,, \qquad \beta > 0$$

when $(z, w) \neq (z_j, w_j)$ lies on E. Then

$$|h(p_j, p)| \geqslant Ma^l\left(\sqrt{|z-z_j|^2 + |w-w_j|^2}\right)^l\cdot\left(\frac{1}{2}\right)^k\cdot \\ \cdot\left(\sqrt{|z-z_j|^2 + |w-w_j|^2}\right)^k\cdot\beta^k = Ma^l\cdot\left(\frac{\beta}{2}\right)^k\cdot|p_j p|^{l+k},$$

Put $Ma^l\left(\frac{\beta}{2}\right)^k = c$, $l+k = a > 0$. For $U\{|z-z_j| < \delta < r, |w-w_j| < s\}$ is $|h(p_j, p)| \geqslant c|p_j p|^a$. Suppose that m points of the system $p^{(n)}$ lie in U. We denote this points by $p_{k_1}, \ldots, p_{k_m}$. Since the function $h(p_j, p)$ is continuous on E and $h(p_j, p) = 0$ only for $p = p_j$, $p \in E$ there exists a positive number d such that $|h(p_j, p_{k_{m+v}})| > d$, $v = 1, 2, \ldots, n-m$.

Therefore

$$\Delta_n(E) = \prod_{v=1}^{m} |h(p_j, p_{k_v})| \cdot \prod_{v=1}^{n-m} |h(p_j, p_k)| \geqslant \prod_{v=1}^{m} |p_j p_k|^a \cdot c^m \cdot d^{n-m} =$$

$$= c^m d^{n-m} \left(\frac{L}{n}\right)^{am} \prod_{k_v \neq j} [j!(m-j)!]^a > C^n \cdot L^{am} \cdot \left(\frac{m}{2en}\right)^{am}, \qquad C = \min(c, d).$$

It follows that

$$\lim_{n\to\infty} \sqrt[n]{\Delta_n(E)} \geqslant C \cdot L^{ab} \cdot \left(\frac{b}{2e}\right)^{ab} > 0, \qquad b = \lim_{n\to\infty} \frac{m}{n}.$$

(For $n \to \infty$ is $m \to \infty$ and $\lim_{n\to\infty} \frac{m}{n} \neq 0$). Therefore is $v_h(E) > 0$.

REFERENCE

[1] F. Leja, *Teoria funkcji analitycznych*, Warszawa 1957.