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On the E. Levi's condition of the analytic convexity of a domain in the space of two complex variables

Let

(1) $$\Phi \equiv \Phi(x_1, y_1, x_2, y_2) = 0$$

be the boundary of a domain D in the space of two complex variables $z_k = x_k + iy_k$ and let $P = P(x_1^0, y_1^0, x_2^0, y_2^0)$ be a fixed point on the hypersurface (1). Suppose that the sufficient small hypersphere with the center at P and radius r is divided by the hypersurface (1) into two parts: one of them is in- and the other outside D. Let Φ be positive outside D. E. Levi introduced the following definition ([2], p. 154). The domain D is *analytic convex (pseudoconvex)* at P if every analytic two-dimensional surface which passes through P contains in every neighbourhood of P points $\neq P$ at which is $\Phi \geqslant 0$.

The hypersphere is an example of a domain which is pseudoconvex at every boundary point P. In fact, let

(2) $$\Phi \equiv x_1^2 + y_1^2 + x_2^2 + y_2^2 - R^2 = 0$$

be the boundary of a hypersphere and $z_1 = f(z_2)$ be an analytic surface which passes through the boundary point P $(R, 0, 0, 0)$. The equation of the surface can be written in the form

(3) $$x_1 = u(x_2, y_2), \qquad y_1 = v(x_2, y_2)$$

where u and v are harmonic functions defined in a neighbourhood $x_2^2 + y_2^2 < r^2$ of the point $x_2 = 0$, $y_2 = 0$.

Suppose first that $u \equiv \text{const}$, $v \equiv \text{const}$. As the point P lies on the surface (3) is $u^2(0, 0) + v^2(0, 0) = R^2$. Along the surface (3) the function (2) is non-negative and therefore the hypersurface (2) satisfies the E. Levi's condition at the point P.

In the general case along the surface (3) is

$$u^2(x_2, y_2) + v^2(x_2, y_2) + x_2^2 + y_2^2 - R^2 = 0$$

where $u^2(0, 0) + v^2(0, 0) = R^2$. From the maximum principle follows that in every neighbourhood of the point $x_2 = 0$, $y_2 = 0$ there exist points such that

$$u^2(x_2, y_2) + v^2(x_2, y_2) > u^2(0, 0) + v^2(0, 0) = R^2.$$

Therefore $\Phi > x_2^2 + y_2^2 > 0$ and the hypersphere is analytic convex at P.

E. Levi proved the following theorem:

The necessary condition for the analytic convexity of a domain D at the boundary point P is

$$L(\Phi) = -\begin{vmatrix} 0, & \Phi_{\bar{z}_1}, & \Phi_{\bar{z}_2} \\ \Phi_{z_1}, & \Phi_{z_1\bar{z}_1}, & \Phi_{z_1\bar{z}_2} \\ \Phi_{z_2}, & \Phi_{z_2\bar{z}_1}, & \Phi_{z_2\bar{z}_2} \end{vmatrix} \geqslant 0$$

at P, where the boundary hypersurface Φ is twice continuously differentiable. The sufficient condition is $L(\Phi) > 0$ at P. If $\Phi = 0$ is an analytic hypersurface then $L(\Phi) \equiv 0$ and vice versa.

We shall give some sufficient conditions for the analytic convexity of a domain D bounded by $\Phi = 0$ at a point P.

Suppose that the equation of the boundary hypersurface is

$$\Phi \equiv x_1 - \varphi(y_1, x_2, y_2) = 0\,. \tag{4}$$

Let $P = P(x_1^0, y_1^0, x_2^0, y_2^0)$ be a fixed point on (4) and let

$$\begin{gathered} u(x_2^0, y_2^0) = x_1^0\,, \qquad v(x_2^0, y_2^0) = y_1^0 \\ u(x_2^0, y_2^0) - \varphi[v(x_2^0, y_2^0), x_2^0, y_2^0] = 0 \end{gathered} \tag{5}$$

be for an analytic surface which passes through P. Suppose that $u(x_2, y_2) \equiv \text{const}$, $v(x_2, y_2) \equiv \text{const}$ in a neighbourhood of (x_2^0, y_2^0). If the domain D is analytic convex at P, then in every neighbourhood of (x_2^0, y_2^0) there exist points such that

$$u(x_2, y_2) - \varphi[v(x_2, y_2), x_2, y_2] \equiv u(x_2^0, y_2^0) - \varphi[v(x_2^0, y_2^0), x_2, y_2] \geqslant 0\,. \tag{6}$$

We obtained the following sufficient condition for the analytic convexity of D at P (with respect to the class of surfaces $u \equiv \text{const}$, $v \equiv \text{const}$).

If the function $\varphi(y_1, x_2, y_2)$ has the maximum at the point (y_1^0, x_2^0, y_2^0) then the condition (6) is satisfied.

If there exists a neighbourhood of the point (x_2^0, y_2^0) such that

$$\varphi(y_1^0, x_2, y_2) > \varphi(y_1^0, x_2^0, y_2^0)$$

for $x_2, y_2 \neq x_2^0, y_2^0$ (it m. the function φ has a strong minimum at x_2^0, y_2^0) then the domain D is not analytic convex at P.

In the general case (when $u \not\equiv \text{const}$, $v \not\equiv \text{const}$) the Levi's condition can be expressed as follows:

In every neighbourhood of P there exist points $\neq P$ such that

$$u(x_2, y_2) - \varphi[v(x_2, y_2), x_2, y_2] \geqslant 0 \tag{7}$$

for every pair of conjugated harmonic functions which satisfy (5).

The condition (7) can be written in the form

$$u(x_2, y_2) - u(x_2^0, y_2^0) \geqslant \varphi[v(x_2, y_2), x_2, y_2] - \varphi[v(x_2^0, y_2^0), x_0^2, y_0^2]\,. \tag{8}$$

From the maximum principle for the harmonic function $u(x_2, y_2)$ follows that in every neighbourhood of (x_2^0, y_2^0) there exist points such that $u(x_2, y_2) - u(x_2^0, x_2^0) > 0$. Therefore we obtain the following sufficient condition

If the function $\varphi(y_1, x_2, y_2)$ *achieves maximum at the point* (y_1^0, x_2^0, y_2^0), *then the difference* $\varphi[v(x_2, y_2), x_2, y_2] - \varphi[v(x_2^0, y_2^0), x_2^0, y_2^0]$ *is* $\leqslant 0$ *in a neighbourhood of* (x_2^0, y_2^0) *and the condition* (8) *is satisfied.*

Suppose that the hypersurface $x_1 = \varphi(y_1, x_2, y_2)$ is of the class C^2 and that there exists a pair of conjugated harmonic functions $u(x_2, y_2)$, $v(x_2, y_2)$ such that

$$u(x_2, y_2) < \varphi[v(x_2, y_2), x_2, y_2)] \tag{9}$$

in a neighbourhood of the point (x_2^0, y_2^0), $\left(x_1^0 = u(x_2^0, y_2^0),\ y_1^0 = v(x_2^0, y_2^0)\right)$. From (9) follows that the domain D bounded by $x_1 = \varphi(y_1, x_2, y_2)$ is not analytic convex at P.

The difference $\varphi[v(x_2, y_2), x_2, y_2] - u(x_2, y_2) = f$ has a strong minimum at the point (x_2^0, y_2^0). The following conditions are sufficient for the existence of the minimum for the function f at x_2^0, y_2^0

$$\begin{gathered} f'_{x_2} = \varphi'_{y_1} \cdot v'_{x_2} + \varphi'_{x_2} - u'_{x_2} = 0\,, \qquad f'_{y_2} = \varphi'_{y_1} \cdot v'_{y_2} + \varphi'_{y_2} - u'_{y_2} = 0\,, \\ f''_{x_2x_2} \cdot f''_{y_2y_2} - (f''_{x_2y_2})^2 > 0\,, \qquad f''_{x_2x_2} > 0\,. \end{gathered} \tag{10}$$

Since $u'_{x_2} = v'_{y_2}$, $u'_{y_2} = -v'_{x_2}$ the conditions (10) can be written in the form

$$\begin{gathered} \varphi'_{y_1} \cdot v'_{x_2} + \varphi'_{x_2} - v'_{y_2} = 0\,, \qquad \varphi'_{y_1} \cdot v'_{y_2} + \varphi'_{y_2} + v'_{x_2} = 0\,, \\ [\varphi''_{y_1y_1} \cdot (v'_{x_2})^2 + 2\varphi''_{y_1x_2} \cdot v'_{x_2} + \varphi'_{y_1} \cdot v''_{x_2x_2} + \varphi''_{x_2x_2} - v''_{x_2y_2}][\varphi''_{y_1y_1} \cdot (v'_{y_2})^2 + \\ + 2\varphi''_{y_1y_2} \cdot v'_{y_2} + \varphi'_{y_1} \cdot v''_{y_2y_2} + \varphi''_{y_2y_2} + v''_{x_2y_2}] - [\varphi''_{y_1y_1} \cdot v'_{x_2} \cdot v'_{y_2} + \varphi''_{y_1y_2} \cdot v'_{x_2} + \\ + \varphi'_{y_1} \cdot v''_{x_2y_2} + \varphi''_{y_1x_2} \cdot v'_{y_2} + \varphi''_{x_2y_2} + v''_{x_2x_2}]^2 > 0\,, \\ \varphi''_{y_1y_1} \cdot (v'_{x_2})^2 + 2\varphi''_{y_1y_2} + \varphi'_{y_1} \cdot v''_{x_2x_2} + \varphi''_{x_2x_2} - v''_{x_2y_2} > 0\,. \end{gathered} \tag{11}$$

Suppose that P is the origin of the coordinate system and that the equation of the hyperplane tangent to $x_1 = \varphi(y_1, x_2, y_2)$ at P is $x_1 = 0$ (it is $\varphi'_{y_1} = \varphi'_{x_2} = \varphi'_{y_2} = 0$ at P). Then from (11) follows

$$\begin{gathered} v'_{x_2} = 0\,, \qquad v'_{y_2} = 0\,, \\ (\varphi''_{x_2x_2} - v''_{x_2y_2})(\varphi''_{y_2y_2} + v''_{x_2y_2}) - (\varphi''_{x_2y_2} + v''_{x_2x_2})^2 > 0\,, \\ \varphi''_{x_2x_2} - v''_{x_2y_2} > 0\,. \end{gathered} \tag{12}$$

Therefore: *If the domain* D *is analytic convex at* P, *then for every harmonic function* $v(x_2, y_2)$, $v(0, 0) = 0$ *is*

$$v'_{x_2} \neq 0 \quad \text{or} \quad v'_{y_2} \neq 0 \quad \text{or} \quad \varphi''_{x_2y_2} - v''_{x_2y_2} \leqslant 0 \quad \text{or} \tag{13}$$

$$(\varphi''_{x_2x_2} - v''_{x_2y_2})(\varphi''_{y_2y_2} + v''_{x_2y_2}) - (\varphi''_{x_2y_2} + v''_{x_2x_2})^2 \leqslant 0\,. \tag{14}$$

We obtained a necessary condition for the analytic convexity of D bounded by the hypersurface $x_1 = \varphi(y_1, x_2, y_2)$ of the class C^2 at P.

The conditions (13) are not satisfied if $v = \text{const}$ or $v''_{x_2y_2} < \varphi''_{x_2y_2}$. The condition (14) is satisfied if

$$(\varphi''_{x_2x_2} - \varphi''_{y_2y_2})^2 + 4(\varphi''_{x_2x_2} \cdot \varphi''_{y_2y_2}) \leqslant 0\,.$$

From the last inequality follows

$$\varphi''_{x_2x_2} + \varphi''_{y_2y_2} = 0\,.$$

Remark. Suppose that the point P is the origin of the coordinate system and that the hypersurface through P lies below the hyperplain $x_1 = 0$, it m. for the points which lie on $x_1 = \varphi(y_1, x_2, y_2)$ and inside D is $x_1 \leqslant 0$. Then the domain D is analytic convex at P. From the maximum principle follows that for every analytic surface $z_1 = f(z_2)$ through P there exists a point for which $x_1 > 0$ (which lies outside the domain D).

REFERENCES

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