ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
Nr 55 PRACE MATEMATYCZNE, ZESZYT 7 1962

Jan Bochenek

On a certain question for the linear combinations of the eigenfunctions in the Sturm-Liouville problem

In this paper a generalization will be given of a certain proposition [1] about zero points of the linear combinations of eigenfunctions.

1. Let us consider the problem of eigenfunctions for the equation:

(1) $$L[u]+\lambda\varrho(x)u=0\,,$$

where

$$L[u]=[p(x)u'(x)]'-q(x)u(x)\,,$$

with boundary conditions

(1a) $$\alpha_1 u(a)-\alpha_2 u'(a)=0\,,\qquad \beta_1 u(b)+\beta_2 u'(b)=0$$

at the end points of the interval

(2) $$a\leqslant x\leqslant b \quad \text{where} \quad a<b\,,$$

α_1, α_2, β_1, β_2 being constant non-negative numbers fulfilling the conditions

$$\alpha_1^2+\alpha_2^2>0 \quad \text{and} \quad \beta_1^2+\beta_1^2>0\,.$$

Under the assumption that the functions $\varrho(x)>0$, $q(x)>0$, $p(x)>0$ are continuous in the interval (2), there exists, as known [2], for the problem (1), (1a) a sequence of eigenvalues

(3) $$0\leqslant\lambda_1<\lambda_2<\lambda_3<\ldots \qquad \lim_{n/\infty}\lambda_n=\infty$$

and [3] a sequence of eigenfunctions of class C^1 in the interval (2)

(4) $$u_1(x),\,u_2(x),\,u_3(x),\,\ldots$$

Moreover, the function $u_n(x)$ has in the interval (a,b) exactly $n-1$ zero points, which are not zero points of its derivative.

We shall establish the following:

Theorem. *Every linear combination of eigenfunctions* (4) *of the form*

(5) $$f(x)=c_m u_m(x)+\ldots+c_n u_n(x)\,,\qquad n\geqslant m\geqslant 1\,,$$

$c_m,\ldots,c_n$ *real constants*, $c_m^2+\ldots+c_n^2>0$, *has in the interval* (a,b) *at least* $m-1$, *and at most* $n-1$ *zero points.*

2. To prove this theorem we shall prove the following lemmas:

Lemma 1. *If:*

1° $f(x)$ *and* $g(x) = p(x)f'(x)$ *(where* $p(x)$ *— continuous and positive in* $\langle a, b\rangle$*) are of class* C^1 *in* $\langle a, b\rangle$,

2° $f(x)$ *fulfills the boundary conditions* (1a) *for* $\alpha_2 = 0$ *or* $\beta_2 = 0$,

3° $f(x) \neq 0$ *in* (a, b),

then there exists a point $\eta \in (a, b)$ *such that* $h(\eta)\cdot f(\eta) < 0$, *where*

$$h(x) = L[f(x)]\,. \tag{6}$$

Proof. We establish the proof in the case when $\beta_2 = 0$ (the proof in the case $\alpha_2 = 0$ is analogous). If $\beta_2 = 0$, then $f(b) = 0$ by (1a). Assuming that $f(x) > 0$, then $f(x)$ in the point a fulfills one of the following conditions: 1° $f(a) = 0$, 2° $f(a) > 0$ and $f'(a) > 0$, 3° $f(a) > 0$ and $f'(a) = 0$.

It is easy to see that in every one of the cases 1°, 2°, 3° there exists a point $a \leqslant x_0 \leqslant b$ such that $f'(x_0) = 0$ and $f(x_0) > 0$. By $f(x_0) > 0$ and $f(b) = 0$, there exists a point $x_0 < x_1 < b$ such that $f'(x_1) < 0$, or else $g(x_0) = p(x_0)f'(x_0) = 0$, $g(x_1) = p(x_1)f'(x_1) < 0$. Hence

$$\frac{g(x_1) - g(x_0)}{x_1 - x_0} < 0\,.$$

But from the mean value theorem

$$\frac{g(x_1) - g(x_0)}{x_1 - x_0} = g'(\eta)\,,$$

hence $g'(\eta) < 0$ and $f(\eta) > 0$, because $x_0 < \eta < x_1 < b$. We have then $h(\eta) = g'(\eta) - q(\eta)f(\eta) < 0$. The inequality $h(\eta) > 0$, when $f(x) < 0$ in (a, b), can be obtained arguing as above, in application to the function $-f(x)$, which gives $-h(\eta) < 0$, and this proves lemma 1.

Lemma 2. *If the functions* $f(x)$ *and* $g(x) = p(x)f'(x)$ *(where* $p(x)$ *is continuous and positive in* $\langle a, b\rangle$*) are of the class* C^1 *in* $\langle a, b\rangle$, *and if for certain points* η_1, y, η_2 *of the interval* (a, b) *the relations* $\eta_1 < y < \eta_2$, $h(\eta_1)f(\eta_1) < 0$, $h(\eta_2)f(\eta_2) < 0$, $f(y) = 0$ *hold, then there exists at least one point* $\xi \in (\eta_1, \eta_2)$ *such that* $h(\xi) = 0$, *where* $h(x)$ *is defined by formula* (6).

Proof. If $f(\eta_1) > 0$ and $f(\eta_2) < 0$ or inversely, then $h(\eta_1)h(\eta_2) < 0$ and the assertion of lemma 2 follows immediately from the continuity of the function $h(x)$. If $f(\eta_1) > 0$ and $f(\eta_2) > 0$ ($f(\eta_1) < 0$ and $f(\eta_2) < 0$), then there exists a point $\eta_1 < x_0 < \eta_2$, in which $f(x)$ attains a local non-positive minimum (a local non-negative maximum). Let us suppose that $f(x)$ attains in x_0 a local non-positive minimum (the proof is analogous for the case of a non-negative maximum), or else $f(x_0) \leqslant 0$, $f'(x_0) = 0$, hence $g(x_0) = p(x_0)f'(x_0) = 0$. Let $\{x_k\}$, $k = 1, 2, \ldots$ be a sequence of points of the interval (a, b) with the properties:

$$\lim_{k/\infty} x_k = x_0 \qquad x_k > x_0 \qquad f'(x_k) \geqslant 0 \qquad (k = 1, 2, \ldots)\,.$$

Such a sequence exists evidently, because $f(x)$ attains a minimum in x_0. As $g(x_k) \geqslant 0$, then

$$\frac{g(x_k)-g(x_0)}{x_k-x_0} \geqslant 0,$$

hence

$$\lim_{k/\infty} \frac{g(x_k)-g(x_0)}{x_k-x_0} = g'(x_0) \geqslant 0$$

and

$$h(x_0) = g'(x_0) - q(x_0) f(x_0) \geqslant 0 . \tag{7}$$

The assertion of lemma 2 follows from (7) (because $h(\eta_1) < 0$ and $h(\eta_2) < 0$ as assumed).

Lemma 3. *If:*

1° *$f(x)$ and $g(x) = p(x)f'(x)$ ($p(x)$ — continuous and positive in $\langle a, b\rangle$) are of class C^1 in $\langle a, b\rangle$,*

2° *$f(x)$ satisfies the boundary conditions* (1a),

3° *$f(x)$ has p zero points in (a, b) $(1 \leqslant p \leqslant \infty)$,*

then the function $h(x)$ defined by the formula (6) *has at least p zero points in this interval.*

Proof. Let us assume for the present that $1 \leqslant p < \infty$ and $x_1, \ldots, x_p$ are the zero points of function $f(x)$ in (a, b). By lemma 1, in every interval

$$(x_i, x_{i+1}) \qquad i = 0, \ldots, p \qquad x_0 = a, \qquad x_{p+1} = b$$

there exists a point η_i such that $h(\eta_i)f(\eta_i) < 0$ $(i = 0, \ldots, p)$, and by lemma 2 in every interval

$$(\eta_i, \eta_{i+1}) \qquad (i = 1, \ldots, p)$$

there exists at least one point ξ_i such that $h(\xi_i) = 0$ $(i = 1, \ldots, p)$, and this proves lemma 3.

If $p = +\infty$, then either $f(x) \equiv 0$ in $\langle a_1, b_1\rangle$ $(a \leqslant a_1 < b_1 \leqslant b)$, and $h(x) \equiv 0$ in $\langle a_1, b_1\rangle$, whence lemma 3 is evident, or $f(x) \not\equiv 0$ in every sub-interval of the interval $\langle a, b\rangle$, and then there exists a denumerable sequence of disjoint intervals $\{(a_n, b_n)\}$ $(n = 1, 2, \ldots)$ such that $f(x) \neq 0$ in (a_n, b_n) and similarly to the case $1 \leqslant p < \infty$, we can show a sequence of points $\{\xi_n\}$ $(n = 1, 2, \ldots)$ such that $h(\xi_n) = 0$ $(n = 1, 2, \ldots)$.

Lemma 4. *If:*

1° *the functions of the sequence $\{f_n(x)\}$ $(n = 1, 2, \ldots)$ are of the class C^1 in the interval $\langle a, b\rangle$,*

2° *$f_n(x)$ tends uniformly to $f(x)$ and $f_n'(x)$ tends uniformly to $f'(x)$ in the interval $\langle a, b\rangle$,*

3° *$f(x)$ has p zeros in the interval $\langle a, b\rangle$ which are not zero points of $f'(x)$, when $f(a) \neq 0$ and $f(b) \neq 0$ $(p < \infty)$, then the number of zeros of every function $f_n(x)$ for sufficiently large n is equal to p in the interval $\langle a, b\rangle$.*

Proof. Let us suppose that

$$(8) \qquad x_1, \ldots, x_p$$

are zero points of the function $f(x)$ in the interval $\langle a, b\rangle$. We put $K = \min\{|f'(x_1)|, |f'(x_2)|, \ldots, |f'(x_p)|\}$. Evidently $K > 0$. From the continuity of $f'(x)$ in $\langle a, b\rangle$ it follows that for every x_i $(i = 1, \ldots, p)$ there exists an interval Δ_i $(\Delta_i \subset \langle a, b\rangle)$ containing x_i such that $|f'(x)| \geqslant \frac{1}{2}K$ (it can be assumed that Δ_i are disjoint). Let us choose for the number γK (by the uniform continuity of $\{f_n(x)\}$ and $\{f_n'(x)\}$) such an N that

$$|f_n(x) - f(x)| \leqslant \gamma K \quad \text{for} \quad n > N$$

and

$$|f_n'(x) - f'(x)| \leqslant \gamma K \quad \text{for} \quad n > N$$

where $0 < \gamma < \frac{1}{2}$ is chosen so that the common part of the set

$$E = \{(x, y) | a \leqslant x \leqslant b, f(x) - \gamma K \leqslant y \leqslant f(x) + \gamma K\}$$

and the interval $\langle a, b\rangle$ would be the union of p intervals $\Delta_i' \subset \Delta_i$ $(i = 1, \ldots, p)$. Now it is evident that in every one of the intervals Δ_i' there exists at least one zero point of the function $f_n(x)$ for $n > N$. We shall show that in none of these intervals there can be more than one zero point of the function $f_n(x)$. For this purpose it is sufficient to prove that for $x \in \Delta_i'$ $(i = 1, \ldots, p)$ and $n > N$ $|f_n'(x)| > 0$. Indeed

$$|f_n'(x)| = |f_n'(x) - f'(x) - f'(x)| \geqslant |f'(x)| - |f_n'(x) - f'(x)| \geqslant \tfrac{1}{2}K - \gamma K > 0 .$$

3. **Proof of the theorem.** It is easy to see that without loss of generality it can be assumed $c_m \neq 0$ and $c_n \neq 0$. For $n = 1$ the theorem is evident. Let us suppose that for $n > 1$ the function defined by formula (5) has p zero points in the interval (a, b). From the definition of the function $f(x)$ it follows that it satisfies the conditions of lemma 3. Applying this lemma we get

$$(9) \qquad h(x) = L[f(x)] = -\lambda_n \varrho(x)\left[\frac{\lambda_m}{\lambda_n} c_m u_m(x) + \frac{\lambda_{m+1}}{\lambda_n} c_{m+1} u_{m+1}(x) + \ldots + c_n u_n(x)\right],$$

where $h(x)$ has at least p zero points in the interval (a, b). As $\lambda_n \neq 0$ (for $n > 1$) and $\varrho(x) \neq 0$ in (a, b), therefore the function

$$(10) \qquad f_1(x) = \frac{\lambda_m}{\lambda_n} c_m u_m(x) + \ldots + c_n u_n(x)$$

has at least p zero points in the interval (a, b). But $f_1(x)$ has the same form as $f(x)$ and satisfies the assumptions of lemma 3. Applying again lemma 3 to $f_1(x)$ we obtain the function

$$(11) \qquad f_2(x) = \left(\frac{\lambda_m}{\lambda_n}\right)^2 c_m u_m(x) + \left(\frac{\lambda_{m+1}}{\lambda_n}\right)^2 c_{m+1} u_{m+1}(x) + \ldots + c_n u_n(x) .$$