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On the functional equation $f(x) = \max\limits_{0\leqslant y\leqslant x}\big(F[g(y), h(x-y)] + f[a(y)+b(x-y)]\big)$

The subject of this paper is the functional equation

$$f(x) = \max_{0\leqslant y\leqslant x}\big(F[g(y), h(x-y)] + f[a(y)+b(x-y)]\big), \tag{1}$$

where $f(x)$ denotes the unknown function, the remaining being given. I shall prove that under certain assumptions on the given functions equation (1) has exactly one solution. Equation (1) is a generalization of the equation occurring in the theory of dynamic programming (see [1] and [2]) of the form

$$f(x) = \max_{0\leqslant y\leqslant x}\big(g(y) + h(x-y) + f[ay+b(x-y)]\big). \tag{2}$$

For equation (2) under certain assumptions on the given functions $g(y)$ and $h(y)$ and under the assumption that a and b are given numbers satisfying inequality $0 \leqslant a, b < 1$, there is a known theorem on the existence and uniqueness of the solution (see [1]). In this paper we shall prove the following:

Theorem. *If*

a) $g(x)$ *and* $h(x)$ *are continuous for* $x \geqslant 0$ *and* $g(0) = h(0) = 0$;

b) $|F(u, v)| \leqslant |u+v|$, $F(u, v)$ *is defined and continuous for arbitrary* u *and* v, *and* $F(0, 0) = 0$;

c) *if* $m(x) = \max\limits_{0\leqslant y\leqslant x}\max\big(|g(y)|, |h(y)|\big)$ *and* $c(y) = \max\big(a(y), b(y)\big)$,

then $\sum\limits_{n=0}^{\infty} m[c^n(x)] < +\infty$ $\big(c^n(x)$ *is the n-th iteration of function* $c(x)\big)$ *for each* $x \geqslant 0$;

d) $a(y)$ *and* $b(y)$ *are continuous increasing functions and satisfy the conditions*

$$0 \leqslant a(y) < y, \quad 0 \leqslant b(y) < y, \quad c(x-y) \leqslant c(x) - c(y);$$

then for $x \geqslant 0$ *equation* (1) *has exactly one continuous solution such that* $f(0) = 0$.

Proof. The proof will be given by method of successive approximations. We put

$$f_1(x) = \max_{0\leqslant y\leqslant x} F[g(y), h(x-y)],$$

$$f_2(x) = \max_{0\leqslant y\leqslant x}\big(F[g(y), h(x-y)] + f_1[a(y) + b(x-y)]\big)$$

and in general

$$(3) \qquad f_n(x) = \max_{0\leqslant y\leqslant x}\left(F[g(y), h(x-y)] + f_{n-1}[a(y)+b(x-y)]\right).$$

Function $f_1(x)$, as the maximum of the continuous function in a closed interval, is continuous too. For the same reason each function of the sequence $\{f_n(x)\}$ is continuous. To show the convergence of sequence $\{f_n(x)\}$ we shall evaluate the n-th term of the series

$$(4) \qquad \sum_{n=1}^{\infty} |f_n(x) - f_{n+1}(x)|.$$

To shorten the notation let us put

$$(5) \qquad T(f_n, y) = F[g(y), h(x-y)] + f_n[a(y)+b(x-y)].$$

If $y_n(x)$ is a function, for which the right-hand side of (5) reaches the maximum and $0 \leqslant y_n(x) \leqslant x$, then by (3) we may write

$$(6) \qquad f_{n+1}(x) = T(f_n, y_n), \qquad f_{n+2}(x) = T(f_{n+1}, y_{n+1}).$$

There may be more functions $y_n(x)$, for which the right-hand side of (5) reaches the maximum; we take whichever of them. From the extremal property of function $y_n(x)$ it follows the inequality

$$(7) \qquad T(f_n, y_n) \geqslant T(f_n, y_{n+1})$$

and

$$(8) \qquad T(f_{n+1}, y_{n+1}) \geqslant T(f_{n+1}, y_n).$$

From (7) and (8) we get

$$(9) \qquad T(f_n, y_{n+1}) - T(f_{n+1}, y_{n+1}) \leqslant f_{n+1}(x) - f_{n+2}(x) \leqslant T(f_n, y_n) - T(f_{n+1}, y_n).$$

Hence

$$(10) \qquad |f_{n+1}(x) - f_{n+2}(x)| \leqslant \max\left(|T(f_n, y_{n+1}) - T(f_{n+1}, y_{n+1})|,\; |T(f_n, y_n) - T(f_{n+1}, y_n)|\right).$$

From (5) and (10) we get

$$(10') \qquad |f_{n+1}(x) - f_{n+2}(x)| \leqslant \max\left(|f_n[a(y_n)+b(x-y_n)] - f_{n+1}[a(y_n)+b(x-y_n)]|,\; |f_n[a(y_{n+1})+b(x-y_{n+1})] - f_{n+1}[a(y_{n+1})+b(x-y_{n+1})]|\right).$$

We put

$$(11) \qquad u_n(x) = \max_{0\leqslant z\leqslant x} |f_n(z) - f_{n+1}(z)| \qquad (n = 1, 2, \ldots).$$

By (d) it is evidently

$$a(y) + b(x-y) \leqslant c(y) + c(x-y) \leqslant c(y) + c(x) - c(y) = c(x) < x.$$

Hence, from (11) and (10′), we get

(12) $$u_{n+1}(x) \leqslant u_n[c(x)].$$

It remains to evaluate $u_1(x)$.

$$u_1(x) = |f_1(x)-f_2(x)| \leqslant \max\big(|f_1[a(y_1)+b(x-y_1)]|,\ |f_1[a(y_2)+b(x-y_2)]|\big)$$
$$\leqslant \max_{0\leqslant z\leqslant c(x)} |f_1(z)| \leqslant \max_{0\leqslant z\leqslant c(x)} \max_{0\leqslant y\leqslant z} |F[g(y), h(z-y)]|$$
$$\leqslant \max_{0\leqslant z\leqslant c(x)} \max_{0\leqslant y\leqslant z} |g(y)+h(z-y)| \leqslant 2 \max_{0\leqslant z\leqslant c(x)} \max_{0\leqslant y\leqslant z} (|g(y)|, h(y)|) = 2m[c(x)].$$

Therefore

(13) $$u_1(x) \leqslant 2m[c(x)].$$

From (12) and (13) we get

(14) $$u_n(x) \leqslant 2m[c^n(x)].$$

Thus, by (c) the series $\sum_{n=1}^{\infty} u_n(x)$ is convergent, and in each finite interval uniformly convergent, since $m(x)$ is non-decreasing. It follows, therefore, that there exists a function $f(x) = \lim_{n\to\infty} f_n(x)$, continuous and satisfying equation (1) and $f(0) = 0$.

We shall now prove the uniqueness of that solution. Let $F(x)$ be an another solution of equation (1), defined for all $x \geqslant 0$, continuous at zero and $F(0) = 0$. We have

$$f(x) = \max_{0\leqslant y\leqslant x} T(f, y) \quad \text{for some function} \quad 0 \leqslant y(x) \leqslant x,$$

$$F(x) = \max_{0\leqslant w\leqslant x} T(F, w) \quad \text{for some function} \quad 0 \leqslant w(x) \leqslant x.$$

In the same manner as above we get

(15) $$|f(x)-F(x)| \leqslant \max\big(|T(f_n, y)-T(F, y)|,\ |T(f, w)-T(F, w)|\big)$$
$$\leqslant \max\big(|f[a(y)+b(x-y)]-F[a(y)+b(x-y)]|,\ |f[a(w)-b(x-w)]-F[a(w)+ +b(x-w)]|\big).$$

Let us put

(16) $$u(x) = \sup_{0\leqslant z\leqslant x} |f(z)-F(z)|.$$

For $x = 0$, $u(x)$ is continuous, and $u(0) = 0$. From (15) we have

(17) $$u(x) \leqslant u[c(x)].$$

Hence, iterating we get

(18) $$u(x) \leqslant u[c^n(x)] \quad \text{for each} \quad n \geqslant 1.$$

Passing in (18) to the limit as $n \to \infty$, by the continuity of $u(x)$ at zero, condition $u(0) = 0$ and assumption (c), we get

$$u(x) \leqslant 0 \,. \tag{19}$$

It follows from (19) that $u(x) = 0$ for each $x \geqslant 0$, i.e. $f(x)$ is identical with $F(x)$.

Note: The above theorem will also be true under other assumptions on continuous functions $a(y)$ and $b(y)$; e.g. assuming that

$$0 \leqslant a(y) < \frac{y}{2} \quad \text{and} \quad 0 \leqslant b(y) < \frac{y}{2},$$

and defining

$$c(x) = 2 \max_{0 \leqslant y \leqslant x} \max[a(y), b(y)]$$

we get

$$\max_{0 \leqslant y \leqslant x} [a(y) + b(x-y)] \leqslant \max_{0 \leqslant y \leqslant x} 2 \max[a(y), b(y)] = c(x) < x$$

and the remaining part of the proof will remain the same.

REFERENCES

[1] R. Bellman, *Dynamic Programming*, Princeton 1957, Princeton Univ. Press.
[2] A. Vazsenyi, *Scientific Programming in Business and Industry*, New York 1958, J. Wiley.