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On the geometrical interpretation of some dual problems in the theory of linear programming

An impulse to write the present note were some remarks of J. Górski.

1. One of the simplest problems in the theory of linear programming (S. Gass [1]) is the following:

Problem 1. Let be done a matrix $A = \{a_{ij}\}$ $(i = 1, \ldots, m,\ j = 1, \ldots, n)$ and systems of numbers $b_1, \ldots, b_m$ and $c_1, \ldots, c_n$. Let be

$$f(x_1, \ldots, x_n) = \sum_{j=1}^{n} c_j x_j \tag{1.1}$$

and let a set Q be defined by the following conditions:

$$x_j \geqslant 0 \qquad (j = 1, \ldots, n) \tag{1.2}$$

$$\sum_{j=1}^{n} a_{ij} x_j \geqslant b_i \qquad (i = 1, \ldots, m)\,. \tag{1.3}$$

Find a n-dimensional point $(\xi_1, \ldots, \xi_n) \in Q$, such that

$$f(\xi_1, \ldots, \xi_n) = \min_{Q} f(x_1, \ldots, x_n)\,.$$

Remark 1. The set Q defined by the conditions (1.2) and (1.3) is convex. It is possible, in particular, that Q will be empty. In the sequel we shall assume that Q is not empty.

We can give the following geometrical interpretation of this problem. If we denote by Φ the family of hyperplanes

$$\sum_{j=1}^{n} c_j x_j = C \tag{1.4}$$

such that for each $\varphi \in \Phi$, $\varphi \cap Q \neq 0$, then we must find in Φ the hyperplane, for which the number C is the smallest. That hyperplane will be named the *minimal hyperplane* of the family Φ and we shall denote it by $\varphi_{\min}(\Phi)$. Each point $(\xi_1, \ldots, \xi_n) \in \varphi_{\min}(\Phi) \cap Q$ is a solution of the problem 1. From the general theory we know very well that the solutions of the problem 1 are on the boundary of the set Q. The dual problem to this problem 1 is the following

Problem 2. Let be

$$g(y_1, \ldots, y_m) = \sum_{j=1}^{n} b_j y_j \tag{1.5}$$

and let a set $\hat{Q}$ be defined by the following conditions:

$$y_j \geqslant 0 \quad (j = 1, \ldots, m) \tag{1.6}$$

$$\sum_{j=1}^{m} a_{ji} y_j \leqslant c_i \quad (i = 1, \ldots, n)\,. \tag{1.7}$$

Find a m-dimensional point $(\eta_1, \ldots, \eta_m) \in \hat{Q}$, such that

$$g(\eta_1, \ldots, \eta_m) = \max_{\hat{Q}} g(y_1, \ldots, y_m)\,.$$

If we denote by Ψ the family of hyperplanes

$$\sum_{j=1}^{m} b_j y_j = D \tag{1.8}$$

such that for each $\psi \in \Psi$, $\psi \cap \hat{Q} \neq 0$, then we must find in the family Ψ the hyperplane for which the number D is the largest. That hyperplane will be named the *maximal hyperplane* of the family Ψ and we shall denote it by $\psi_{\max}(\Psi)$. Each point $(\eta_1, \ldots, \eta_m) \in \psi_{\max}(\Psi) \cap \hat{Q}$ is a solution of the problem 2. It is well known, from the general theory, that the solution of the problem 1 exists if and only if there exists the solution of the problem 2, and moreover if $\xi_1, \ldots, \xi_n$ and $\eta_1, \ldots, \eta_m$ are these solutions, then

$$f(\xi_1, \ldots, \xi_n) = \min_{Q} f(x_1, \ldots, x_n) = g(\eta_1, \ldots, \eta_m) = \max_{\hat{Q}} g(y_1, \ldots, y_m)\,,$$

what means that the number C for the minimal hyperplane of the family Φ is equal the number D for the maximal hyperplane of the family Ψ.

2. We shall assume now that $n = m$ and $b_j > 0$, $c_j > 0$ for all j. Under these assumptions we can write the system of inequalities (1.3) in the following form

$$\sum_{j=1}^{n} d_{ij} x_j \geqslant 1 \quad (i = 1, \ldots, n)\,, \tag{2.1}$$

where $d_{ij} = a_{ij}/b_i$. Moreover, after a simple transformation $z_i = c_i x_i$ we can replace the problem 1 by the following

Problem 1a. Let

$$h(x_1, \ldots, x_n) = \sum_{j=1}^{n} x_i \tag{2.2}$$

and let a set R be defined by the following conditions

$$x_i \geqslant 0 \quad (i = 1, \ldots, n) \tag{2.3}$$

$$\sum_{j=1}^{n} k_{ij} x_j \geqslant 1 \quad (i = 1, \ldots, n). \tag{2.4}$$

Find a n-dimensional point $(\xi_1, \ldots, \xi_n) \in R$ such that

$$h(\xi_1, \ldots, \xi_n) = \min_{R} h(x_1, \ldots, x_n).$$

The dual problem to the problem 1a is the following

Problem 2a. Let

$$h(y_1, \ldots, y_n) = \sum_{j=1}^{n} y_i \tag{2.5}$$

and let a set $\hat{R}$ be defined by the following conditions:

$$x_i \geqslant 0 \quad (i = 1, \ldots, n) \tag{2.3}$$

$$\sum_{j=1}^{n} k_{ji} x_j \leqslant 1 \quad (i = 1, \ldots, n). \tag{2.6}$$

Find a n-dimensional point $(\eta_1, \ldots, \eta_n) \in \hat{R}$ such that

$$h(\eta_1, \ldots, \eta_n) = \max_{\hat{R}} h(y_1, \ldots, y_n).$$

Remark 2. Because $n = m$, we can consider the both problems in the same space. Moreover we can identify the functions (2.2) and (2.5) and we can denote these both functions by the same letter h. It follows from the theorem quoted in the first paragraph that a solution of the problem 1a exists if and only if there exists a solution of the problem 2a and $\min\limits_{R} h = \max\limits_{\hat{R}} h$. It is easy to see that in this case the equations (1.4) and (1.8) have the same form. Solutions of the problems 1a and 2a are on the same hyperplane defined by the equation:

$$\sum_{i=1}^{n} x_i = L \quad (L = \min_{R} h = \max_{\hat{R}} h). \tag{2.7}$$

That hyperplane is the unique hyperplane among the hyperplanes

$$\sum_{i=1}^{n} x_i = \text{const} \tag{2.8}$$

which separates the interiors of the sets R and $\hat{R}$.