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On the extremal solutions of a functional equation

In the present paper we shall consider the problem of the existence of the maximal and the minimal solutions of the functional equation:

$$y(x) = F\{x, y[f(x)]\}\,.$$

1. We introduce the following definitions (G. Birkhoff [1]):

Definition 1. We call a set P *partly ordered set* if for some pairs of elements $x, y \in P$, a relation $x \leqslant y$ is defined, in such a way that

(a) for each $x \in P$, $x \leqslant x$,
(b) if $x \leqslant y$ and $y \leqslant x$, then $x = y$,
(c) if $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$.

Definition 2. Let Q be a partly ordered set and let $Q \subset P$. We call z *upper bound of Q in P*, if

(d) $z \in P$,
(e) $x \in Q \Rightarrow x \leqslant z$.

Remark 1. On the analogous way we define the *lower bound of Q in P*.

Definition 3. We call $\hat{z}$ *supremum of Q (sup Q)* if $\hat{z}$ is the upper bound of Q in P and

(f) x is the upper bound of Q in $P \Rightarrow \hat{z} \leqslant x$.

Remark 2. On the analogous way we define *infimum of Q (inf Q)*.

Remark 3. It is easy to see that every partly ordered set can have at most one supremum and at most one infimum.

Definition 4. Let the function $V(x)$ be defined in a partly ordered set P and let $V(P) \subset R$, where R is some partly ordered set. We call $V(x)$ *increasing* if:

(g) $x \leqslant y \Rightarrow V(x) \leqslant V(y)$.

Remark 4. On the analogous way we define the *decreasing* function.

It is possible to prove the following theorems (A. Pelczar [2]).

Theorem T_1. *If:*

I. *P is a not empty and partly ordered set,*
II. *for each not empty subset $Q \subset P$, there exists sup Q in P,*
III. *$V(z)$ is a function defined in P,*

IV. $V(P) \subset P$,
V. *$V(z)$ is increasing in P,*
VI. *there exists in P a point z_0 such that $z_0 \leqslant V(z_0)$,*
then the equation

$$z = V(z) \tag{1.1}$$

has solutions in P, and moreover among them there exists a maximal solution z^*, (i.e. for each solution x of the equation (1.1) *is $x \leqslant z^*$).*

Theorem T_2. *If we replace the assumptions II and VI of the theorem T_1 by the following assumptions:*

II′. *for each not empty subset $Q \subset P$, there exists inf Q in P,*
VI′. *there exists in P a pont z_1 such that $V(z_1) \leqslant z_1$,*
leaving the other assumptions of the theorem T_1 without any changes, then the equation (1.1) *has the solutions in P, and moreover among them there exists the minimal solution z^*.*

Theorem T_3. *If the set P and the function $V(x)$ fulfill the assumptions I—V of the theorem T_1 and if for $y \epsilon P$:*

$$y \leqslant V(y) \tag{1.2}$$

then

$$y \leqslant z^* \tag{1.3}$$

where z^* is the maximal solution of the equation (1.1).

2. We shall prove the following

Theorem 1. *If*:

1. *$F(x, y)$ is a function defined in the set $\{\langle a, b\rangle \times (-\infty, \infty)\}$*
2. $|F(x, y)| \leqslant C$
3. $|F(x, y) - F(\bar{x}, \bar{y})| \leqslant M|x-\bar{x}| + K|y-\bar{y}|$
4. *if $y \leqslant \bar{y}$, then $F(x; y) \leqslant F(x, \bar{y})$*
5. *$f(x)$ is a function defined in the interval $\langle a, b\rangle$*
6. $f(\langle a, b\rangle) \subset \langle a, b\rangle$
7. *$|f(x) - f(\bar{x})| \leqslant N|x-\bar{x}|$, where N is a number such that $K \cdot N < 1$,*

then the equation

$$y(x) = F\{x, y[f(x)]\} \tag{2.1}$$

has, in the class of functions fulfilling the Lipschitz condition in the interval $\langle a, b\rangle$ with the constant $L = M/(1-KN)$, the maximal and the minimal solutions.

Proof. Let P be the class of the functions fulfilling the Lipschitz condition in the interval $\langle a, b\rangle$ with the constant $L = M/(1-KN)$. We introduce the relation $\varphi \leqslant \psi$, for $\varphi, \psi \epsilon P$ in such a way that: $\varphi \epsilon \psi \equiv \varphi(x) \leqslant \psi(x)$ for each $x \epsilon \langle a, b\rangle$. The set P, with this relation $\leqslant$ is a partly ordered set in the sense of the definition 1. It is easy to see that for each $Q \subset P$, supremum and infimum in the sense of the definition 3, are supremum and infimum of the set Q