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On zero points of the linear combinations of the eigenfunctions in the Sturm-Liouville problem

In this paper a generalization will be given of a certain proposition [1] about zero points of the linear combinations of eigenfunctions.

1. Let us consider the problem of eigenfunctions for the equation

$$L[u]+\lambda\varrho(x)u=0, \tag{1}$$

where

$$L[u]=[p(x)u'(x)]'-q(x)u(x),$$

with boundary conditions

$$\alpha_1 u(a)-\alpha_2 u'(a)=0, \quad \beta_1 u(b)+\beta_2 u'(b)=0 \tag{2}$$

at the end points of the interval $\langle a,b\rangle$. α_1, α_2, β_1, β_2 are real numbers fulfilling the conditions: $\alpha_1^2+\alpha_2^2>0$ and $\beta_1^2+\beta_2^2>0$. Under the assumption that the functions $\varrho(x)>0$, $q(x)\geqslant 0$ and $p(x)>0$ are continuous in the interval $\langle a,b\rangle$ there exists [2] for the problem (1), (2) a sequence of eigenvalues

$$0\leqslant\lambda_1<\lambda_2<\lambda_3,\ \ldots,\ \lim_{n\to\infty}\lambda_n=\infty$$

and [3] a sequence of eigenfunctions of class C^1 in the interval $\langle a,b\rangle$

$$u_1(x),\ u_2(x),\ u_3(x),\ \ldots \tag{3}$$

The function $u_n(x)$ has in the interval (a,b) exactly $n-1$ zero points, which are not zero points of its derivative.

Under the above assumptions in [1] there is given the proof of the following theorem: *Every linear combination of eigenfunctions* (3) *of the form*

$$f(x)=c_m u_m(x)+\ldots+c_n u_n(x), \quad n\geqslant m\geqslant 1, \tag{4}$$

$c_m,\ldots,c_n$ *real constants*, $c_m^2+\ldots+c_n^2>0$, *has in the interval* (a,b) *at least* $m-1$, *and at most* $n-1$ *zero points*. The proof of this theorem is based upon the following lemmas, where we assume that the functions $f(x)$ and $g(x) = p(x)f'(x)$ are of class C^1 in $\langle a, b\rangle$.

Lemma 1. *If $f(x)$ fulfills the boundary conditions (2) for $\alpha_2 = 0$ or $\beta_2 = 0$ and $f(x) \neq 0$ in (a, b), then there exists a point $\zeta \in (a, b)$ such that $h(\zeta)f(\zeta) < 0$, where $h(x) = L[f(x)]$.*

Lemma 2. *If the function $f(x)$ fulfills the boundary conditions (2), $f(y) = 0$ $(a < y < b)$, $f(x) \neq 0$ in (a, c) and in (d, b) $(a < c \leqslant y \leqslant d < b)$, $h(\zeta_1)f(\zeta_1) < 0$ and $h(\zeta_2)f(\zeta_2) < 0$ $(a < \zeta_1 < c, d < \zeta_2 < b)$, then there exists at least one point $\xi \in (\zeta_1, \zeta_2)$ such that $h(\xi) = 0$.*

Lemma 3. *If the function $f(x)$ which fulfills the boundary conditions (2), has p $(1 \leqslant p \leqslant \infty)$ zero points in the interval (a, b), then the function $h(x)$ has in this interval at least p zero points.*

Lemma 4. *If the functions of the sequence $\{f_n(x)\}$ are of class C^1 in the interval $\langle a, b\rangle$, $\lim_{n\to\infty} f_n(x) = f(x)$, $\lim_{n\to\infty} f_n'(x) = f'(x)$ uniformly in $\langle a, b\rangle$, $f(x)$ has p $(p < \infty)$ zero points in the interval $\langle a, b\rangle$ which are not zero points of $f'(x)$ and $f(a) \neq 0$, $f(b) \neq 0$, then for sufficiently large n the number of zero points of every function $f_n(x)$ in the interval (a, b) is equal to p.*

The purpose of the present paper is a generalization of above theorem. Under the additional assumption that $p(x)$, $q(x)$, $\varrho(x)$ are of class C^1 in (a, b) we shall consider zero points of $f(x)$ with its multiplicity: r-tuple zero will be counted as r zeros.

2. We shall speak that a function $f(x)$, of class C^1 in a neighborhood of a point x_0, has in x_0 zero of order N, if

$$f(x_0) = f'(x_0) = \dots = f^{(N-1)}(x_0) = 0\,, \qquad f^{(N)}(x_0) \neq 0\,.$$

Lemma 5. *If the functions $f(x)$ and $h(x) = L[f(x)]$ have the finite number of zero points in the interval (a, b) and $f(x)$, of class C^∞ in (a, b), has in $x_0 \in (a, b)$ zero of order N, $N \geqslant 2$, then*

$$\lim_{x\to x_0} \frac{(x-x_0)^2 h(x)}{f(x)} = N(N-1)p(x_0) > 0\,. \tag{5}$$

The proof of this lemma is immediate.

Lemma 6. *If the functions $f(x)$ and $h(x) = L[f(x)]$ have the finite number of zero points in the interval (a, b), $f(x)$ of class C^∞ in (a, b) has in $x_0 \in (a, b)$ zero of order N $(N \geqslant 1)$ and there exists points ζ_1 and ζ_2 such that $\zeta_1 < x_0 < \zeta_2$ and $h(\zeta_i)f(\zeta_i) < 0$ $(i = 1, 2)$, $f(x) \neq 0$ for $\zeta_1 < x < \zeta_2$ and $x \neq x_0$, then the function $h(x)$ has in the interval (ζ_1, ζ_2) at least N zeros.*

Proof. We shall distinquish three cases: (1) $N = 1$, (2) $N = 2$, (3) $N > 2$. For (1) the thesis of lemma follows from lemma 2. For (2) from (5) it follows, that $h(x)f(x) > 0$ in any neighborhood of point x_0, however at the points ζ_i: $h(\zeta_i)f(\zeta_i) < 0$ $(i = 1, 2)$, hence in the interval (ζ_1, ζ_2) there exists at least two zero points of function $h(x)$. For (3) from (5) it follows that the function $h(x)$ has in x_0 zero of order at least N-2 and $h(x)f(x) > 0$ in any neighborhood of x_0.

Since $h(\zeta_i)f(\zeta_i) < 0$ $(i = 1, 2)$, hence in every interval (ζ_1, x_0) and (x_0, ζ_2) $h(x)$ has at least one zero point.

Lemma 7. *If the function $f(x)$ of class C^∞ in (a, b) satisfies the boundary conditions (2), $f(x)$ has in (a, b) p $(1 \leqslant p < \infty)$ zero points of order $N_1, \ldots, N_p$ respectively, $N_1 + \ldots + N_p = s$, then the function $h(x)$ has in the interval (a, b) at least s zeros.*

Proof. Let us assume that $s < \infty$ and let $x_1 < x_2 < \ldots < x_{p-1} < x_p$ be the zero points of $f(x)$ in the interval (a, b). By lemma 1, in every interval (a, x_1), $(x_1, x_2), \ldots, (x_{p-1}, x_p), (x_p, b)$ there exists a point ζ_i, such that $h(\zeta_i)f(\zeta_i) < 0$ $(i = 1, \ldots, p+1)$. If $h(x)$ has a finite number of zero points in the interval (a, b), then from lemma 6 in every interval (ζ_i, ζ_{i+1}) $h(x)$ has N_i zeros $(i = 1, \ldots, p)$. If $s = \infty$, then from $p < \infty$ it follows that at least one of numbers N_i $(i = 1, \ldots, p)$ is infinite. Let $N_k = \infty$ $(1 \leqslant k \leqslant p)$. Then either $h(x)$ has a infinite number of zero points in the interval (a, b) or $h(x)$ has a finite number of zero points in the interval (a, b) but then from (10) it has at the point x_k zero of an infinite order.

3. Theorem. *Every linear combination of eigenfunctions (3) of the form (4) has in the interval (a, b) at least $m-1$ zero points and at most $n-1$ zeros (r-tuple zero is counted as r zeros).*

Proof. Without loss of generality it can be assumed $c_m \neq 0$ and $c_n \neq 0$. For $n = 1$ the theorem is evident. Let us suppose that for $n > 1$ the function $f(x)$ defined by formula (4) has in the interval (a, b) p zero points of order $N_1, \ldots, N_p$ respectively, $N_1 + \ldots + N_p = s$. By theorem cited in the Section 1 $p \leqslant n-1 < \infty$, hence the function $f(x)$ satisfies the assumptions of lemma 7. From this we get that

$$h(x) = L[f(x)] = -\lambda_n \varrho(x)\left[\frac{\lambda_m}{\lambda_n} c_m u_m(x) + \ldots + c_n u_n(x)\right]$$

has at least s zeros in the interval (a, b). As $\lambda_n \neq 0$ (for $n > 1$), therefore

$$f_1(x) = \left[\frac{\lambda_m}{\lambda_n} c_m u_m(x) + \ldots + c_n u_n(x)\right]$$

has in the interval (a, b) at least s zeros. But $f_1(x)$ has the same form as $f(x)$ and satisfies the assumptions of lemma 7. Applying again this lemma to $f_1(x)$ we obtain the function

$$f_2(x) = \left[\frac{\lambda_m}{\lambda_n}\right]^2 c_m u_m(x) + \left[\frac{\lambda_{m+1}}{\lambda_n}\right]^2 c_{m+1} u_{m+1}(x) + \ldots + c_n u_n(x)\,.$$

Proceeding thus further we obtain an infinite sequence of functions

$$f(x), f_1(x), f_2(x), \ldots \tag{11}$$

for which the number of zeros does not decrease with the increase of the index. Now

$$f_k(x) = \left[\frac{\lambda_m}{\lambda_n}\right]^k c_m u_m(x) + \left[\frac{\lambda_{m+1}}{\lambda_n}\right]^k c_{m+1} u_{m+1}(x) + \ldots + c_n u_n(x)\,. \tag{12}$$

We have $0 < \lambda_s/\lambda_n < 1$ for $s = m, \ldots, n-1$. For fixed n the functions $u_s(x)$, $u_s'(x)$ are uniformly bounded in $\langle a, b\rangle$. Hence the sequence (11) and its limit $c_n u_n(x)$ for $k \to \infty$ satisfy the assumptions of lemma 4. Considering that the function $c_n u_n(x)$ has $n-1$ zeros in the interval (a, b), therefore applying lemma 4 to the sequence (11) and to any closed interval $\langle A, B\rangle$ included in (a, b), for which zero points of the function $u_n(x)$ are interior points, we obtain the inequality $s \leqslant n-1$. The proof of inequality $p \geqslant m-1$ is to be found in [1].

REFERENCES

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[3] E. Kamke, *Differentialgleichungen reeller Funktionen*, New York 1947.