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Remark on a paper of P. K. Zeragia

In his paper [6] P. K. Zeragia considers the non-linear system of parabolic differential equations of the form

$$(1) \quad \frac{\partial u_i}{\partial t} - \Delta u_i = f_i(x_i, t, u_1, \dots, u_n) \quad (i = 1, \dots, n),$$

where $x = (x_1, \dots, x_p)$ is a point in region D , situated in the p -dimensional space R , limited by Lapunov's closed surface S , and the following boundary value problem: to find functions $u_1(x, t), \dots, u_n(x, t)$ which are regular in $\bar{D} \times \langle 0, T \rangle$ and satisfy in $D \times (0, T)$ the system (1) and the boundary conditions:

$$(2) \quad u_i(x, 0) = 0 \quad \text{for } x \in D,$$

$$(3) \quad u_i(x, t) = 0 \quad \text{for } (x, t) \in S \times (0, T), \quad (i = 1, \dots, n).$$

To solve the above problem, the author applies Chaplighin's method [1] assuming that the below conditions are fulfilled:

(A) The functions $f_i(x, t, u_1, \dots, u_n)$ ($i = 1, \dots, n$) are continuous in region $(x, t) \in \bar{D} \times \langle 0, T \rangle$, $-\infty < u_i < +\infty$ ($i = 1, \dots, n$) and the following inequality holds true in that region

$$|f_i(x', t, u'_1, \dots, u'_n) - f_i(x'', t, u''_1, \dots, u''_n)| < K |u'_i - u''_i| + K \sum_{a=1}^p |x'_a - x''_a|,$$

$(i = 1, \dots, n)$

where $K > 0$.

$$(B) \quad f_i(x, t, u'_1, \dots, u'_n) \leq f_i(x, t, u''_1, \dots, u''_n) \quad (i = 1, \dots, n),$$

when

$$u'_j \leq u''_j \quad (j = 1, \dots, n).$$

In obtaining the relations (29) page 263, (30) p. 264, on p. 262 and in the proof of the theorem 1 the author several times applies the inequality (A).

However, from the inequality (A) it results* that the function f_i may depend only on one variable u_i and therefore the system (1) takes the form

$$\frac{\partial u_i}{\partial t} - \Delta u_i = f_i(x, t, u_i) \quad (i = 1, \dots, n).$$

Consequently, we have to deal with the problem previously considered by P. K. Zeragia [4], [5] (under some more restrictive assumptions) and W. Mlak [2] (under different, weaker assumptions). Theorems 1 and 2 on fundamental differential inequalities of Chaplignin type, formulated and proved by P. K. Zeragia, are also true under certain weaker assumptions. Similar more general theorems were formulated by J. Szarski in [3].

In the case of system (1) the theorems 1 and 2 will remain true provided that the following assumptions be satisfied:

$$|f_i(x, t, u'_1, \dots, u'_n) - f_i(x, t, u''_1, \dots, u''_n)| \leq g_i(t, u'_1 - u''_1, \dots, u'_n - u''_n) \quad (i = 1, \dots, n)$$

when

$$u'_j \geq u''_j \quad (j = 1, \dots, n),$$

where the functions $g_i(t, y_1, \dots, y_n)$ ($i = 1, \dots, n$) are continuous, nonnegative for $0 \leq t < T$, $y_j \geq 0$ ($j = 1, \dots, n$) and satisfy the condition

$$g_i(t, y'_1, \dots, y'_n) \leq g_i(t, y''_1, \dots, y''_n),$$

when

$$0 \leq y'_j \leq y''_j \quad (j = 1, \dots, n) \text{ and } 0 \leq y'_i = y''_i,$$

and the unique solution of the system

$$\frac{dy_i}{dt} = g_i(t, y_1, \dots, y_n) \quad (i = 1, \dots, n),$$

starting from point $(0, \dots, 0)$ is $y_i(t) \equiv 0$ ($i = 1, \dots, n$).

In view of the above remarks on (A) it is sufficient to introduce the essentially weaker condition

$$(A^*) \quad |f_i(x', t, u'_1, \dots, u'_n) - f_i(x'', t, u''_1, \dots, u''_n)| \leq K \sum_{i=1}^n (u'_i - u''_i) + K \sum_{q=1}^n |x'_q - x''_q|$$

when

$$u'_j \geq u''_j \quad (j = 1, \dots, n), \quad (i = 1, \dots, n),$$

or some other conditions (i.e. conditions similar to those in [2], generalized for a system).

The condition (A*) does not imply that system (1) is a system of equations each containing only one unknown function.

In case condition (A*) is introduced, certain changes will occur in the proof (see pag. 262) of the properties of the upper $\{v_i^{(m)}(x, t)\}$ ($i = 1, \dots, n$) and lower

* As it was remarked by J. Szarski.

$\{w_i^{(m)}(x, t)\}$ ($i = 1, \dots, n$) sequences of functions, defined for (1) and for the boundary problem (2), (3) as the solutions of equations (25) and (26) i.e.

$$\frac{\partial (v_i^{(m)} - v_i^{(m-1)})}{\partial t} - \Delta (v_i^{(m)} - v_i^{(m-1)}) + K (v_i^{(m)} - v_i^{(m-1)}) + \alpha_i^{(m-1)}(x, t) = 0,$$

$$\frac{\partial (w_i^{(m)} - w_i^{(m-1)})}{\partial t} - \Delta (w_i^{(m)} - w_i^{(m-1)}) + K (w_i^{(m)} - w_i^{(m-1)}) + \alpha_i^{(m-1)}(x, t) = 0,$$

where

$$\alpha_i^{(m-1)}(x, t) \equiv \frac{\partial v_i^{(m-1)}}{\partial t} - \Delta v_i^{(m-1)} - f_i(x, t, v_1^{(m-1)}, \dots, v_n^{(m-1)}) \geq 0,$$

$$\alpha_i^{(m-1)}(x, t) \equiv \frac{\partial w_i^{(m-1)}}{\partial t} - \Delta w_i^{(m-1)} - f_i(x, t, w_1^{(m-1)}, \dots, w_n^{(m-1)}) \leq 0,$$

for $(x, t) \in D \times (0, T)$ ($i = 1, \dots, n$).

In order to prove that $v_i^{(1)}(x, t)$ ($i = 1, \dots, n$) are the upper functions, we apply the previously proved inequality that $v_i^{(1)}(x, t) \leq v_i^{(0)}(x, t)$ ($i = 1, \dots, n$) for $(x, t) \in \bar{D} \times (0, T)$, the condition (B), and the definition of $\alpha_i^{(0)}(x, t)$ ($i = 1, \dots, n$) in the following way:

$$\begin{aligned} & \frac{\partial v_i^{(1)}}{\partial t} - \Delta v_i^{(1)} - f_i(x, t, v_1^{(1)}, \dots, v_n^{(1)}) = \\ &= \frac{\partial v_i^{(0)}}{\partial t} - \Delta v_i^{(0)} - \alpha_i^{(0)} - K (v_i^{(1)} - v_i^{(0)}) - f_i(x, t, v_1^{(1)}, \dots, v_n^{(1)}) + \\ & \quad + f_i(x, t, v_1^{(0)}, \dots, v_n^{(0)}) - f_i(x, t, v_1^{(1)}, \dots, v_n^{(1)}) = \\ &= \left[\frac{\partial v_i^{(0)}}{\partial t} - \Delta v_i^{(0)} - f_i(x, t, v_1^{(0)}, \dots, v_n^{(0)}) - \alpha_i^{(0)} \right] - K (v_i^{(1)} - v_i^{(0)}) + \\ & \quad + [f_i(x, t, v_1^{(0)}, \dots, v_n^{(0)}) - f_i(x, t, v_1^{(1)}, \dots, v_n^{(1)})] \geq 0 \quad (i = 1, \dots, n). \end{aligned}$$

Consequently, from theorem 1 it follows that $v_i^{(1)}(x, t) \geq u_i(x, t)$ ($i = 1, \dots, n$) for $(x, t) \in \bar{D} \times (0, T)$ where $u_i(x, t)$ ($i = 1, \dots, n$) is the solution of system (1) and the boundary problem (2), (3) or $v_i^{(1)}(x, t)$ ($i = 1, \dots, n$) are the upper functions.

For the purpose of obtaining required estimates for the difference $v_i^{(m)}(x, t) - w_i^{(m)}(x, t)$ ($i = 1, \dots, n$) it is necessary to consider $\max_{i=1, \dots, n} [v_i^{(m)} - w_i^{(m)}]$ in (29)

p. 263.

Hence, we obtain that

$$v_i^{(m)} - w_i^{(m)} \leq \max_{i=1, \dots, n} [v_i^{(m)} - w_i^{(m)}] \leq L \frac{[(n+1)Kt]^m}{m!},$$

where

$$v_i^{(0)}(x, t) - w_i^{(0)}(x, t) \leq L_i \quad \text{and} \quad L = \max_{i=1, \dots, n} L_i$$

for

$$(x, t) \in \bar{D} \times \langle 0, T \rangle.$$

The same reasoning ought to be followed in proving the uniqueness of the obtained solution (i.e. in (30) p. 264).

REFERENCES

- [1] С. А. Чаплыгин, *Новый метод приближенного интегрирования дифференциальных уравнений*, Москва — Ленинград, 1950.
- [2] W. Mlak, *Parabolic differential inequalities and Chaplignin's method*, Ann. Polon. Math. 8 (1960), p. 139—153.
- [3] J. Szarski, *Sur un système non linéaire d'inégalités différentielles paraboliques* (to appear in Ann. Polon. Math.).
- [4] П. К. Зерагия, *Решение основных граничных задач для нелинейного дифференциального уравнения параболического типа методом акад. С. А. Чаплыгина*, Сообщения АНГССР 17 (1956), p. 103—110.
- [5] П. К. Зерагия, *Граничные задачи для некоторых нелинейных уравнений параболического типа*, Труды Тбилисского Матем. Института АНГССР 24 (1957), p. 195—222.
- [6] П. К. Зерагия, *Решение основной граничной задачи для одной системы дифференциальных уравнений параболического типа методом Чаплыгина* Сообщения АНГССР 26 (1961), p. 257—264.