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On the asymptotic expansion of functions

Denote

$$p_n(x) = a_0 + a_1x + a_2x + \ldots + a_nx^n \qquad n = 0, 1, \ldots$$

We say that a series

$$\sum_{i=0}^{\infty} a_i x^i \tag{1}$$

is an asymptotic expansion of a function $f(x)$ defined for $x > 0$ if

$$\lim_{x\to 0} f(x) = a_0, \quad \lim_{x\to 0} \frac{f(x) - p_{n-1}(x)}{x^n} = a_n \qquad n = 1, 2, \ldots \tag{2}$$

In the present paper we give the solution of the following problem of prof. T. Ważewski: Let (1) be an asymptotic expansion of a function $f(x)$. Does there exist two regular functions $h(x)$ and $g(x)$, $g(x) < f(x) < h(x)$, which have the same asymptotic expansion? We shall prove the following

Theorem. *If a function $f(x)$ has the asymptotic expansion* (1), *then there are two functions $h(x)$ and $g(x)$ and a number a such that $g(x) < f(x) < g(x)$, $g(x)$ and $h(x)$ are of class C^1 in the interval $\langle 0, a\rangle$ and have the asymptotic expansion* (1).

Proof. From (2) for sufficiently small $x_n^{(1)}$ it follows

$$\frac{f(x) - p_{n-1}(x)}{x^n} < a_n + 1, \qquad 0 < x < x_n^{(1)}. \tag{3}$$

Put

$$w_n(x) = p_n(x) + x^n.$$

From (3) we have

$$f(x) < w_n(x) \qquad \text{for} \qquad 0 < x < x_n^{(1)}. \tag{4}$$

Since

$$w_n(x) - w_{n+1}(x) = x^n\big(1 - (a_{n+1} + 1)x\big)$$

for sufficiently small $x_n^{(2)}$

$$w_n(x)-w_{n+1}(x)>0 \quad \text{for} \quad 0<x<x_n^{(2)} .$$

Likewise

$$w_n'(x)-w_{n+1}'(x)>0 \quad \text{for} \quad 0<x<x_n^{(3)} \tag{5}$$

and

$$w_n''(x)-w_{n+1}''(x)>0 \quad \text{for} \quad 0<x<x_n^{(4)} \tag{6}$$

when $x_n^{(3)}$ and $x_n^{(4)}$ are sufficiently small. There exists $x_n^{(5)}$ such that

$$a_1-\frac{1}{2n}<w_n(x)<a_1+\frac{1}{2n} \quad \text{for} \quad 0<x<x_n^{(5)} . \tag{7}$$

We define

$$x_n^* = \min(x_{n+1}^{(1)}, x_n^{(2)}, x_n^{(3)}, x_n^{(4)}, x_{n+1}^{(5)}) .$$

Let be

$$0<x_n<x_n^*, \quad y_n(x, x_n) = w_n'(x_n)-w_{n+1}'(x_n)(x-x_n)+w_n(x_n)-w_{n+1}(x) .$$

The straight line $T_n(x_n)$ given by the equation $y = y_n(x, x_n)$ is tangent to the curve $y = w_n(x)-w_{n+1}(x)$ at the point x_n.

From (6) it follows that the polynomial $y = w_n(x)-w_{n+1}(x)$ is convex in the interval $\langle 0, x_n^{(4)}\rangle$. In the interval $\langle 0, x_n\rangle$ the tangent $T_n(x_n)$ lies therefore below the graph of this polynomial. By $\varphi_n(x, x_n)$ we denote the function whose graph (Fig. 1) in the interval $\langle 0, x_n\rangle$ consists of the segment $\langle 0, x_n^0\rangle$ of x-axis, of a segment of circle tangent to the x-axis and to the tangent $T_n(x_n)$ and of a segment of $T_n(x_n)$. We do this so that

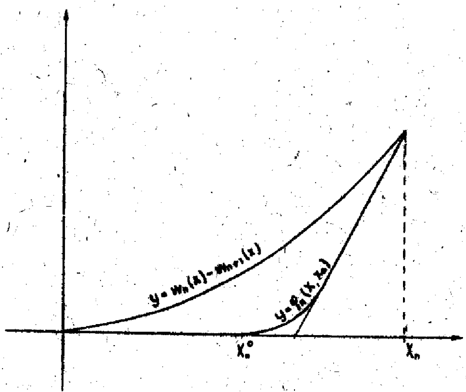

Fig. 1

$$\varphi_n(x, x_n) < w_n(x)-w_{n+1}(x) \quad \text{for} \quad 0<x<x_n .$$

From definition it follows that $f_n(x, x_n)$ is of class C^1. Moreover

$$0<\varphi_n'(x, x_n) \leqslant \varphi_n'(x_n, x_n) = w_n'(x_n)-w_{n+1}'(x_n) . \tag{8}$$

We define the sequence $\{x_n\}$ as follows

$$x_3 = \min(x_1^*, x_2^*, x_3^*), \quad x_n = \frac{1}{n}\min(x_n^*, x_{n-1}^0) \qquad n = 4, 5, \ldots$$

It is easily to see that

$$0 < x_n < x_{n-1}, \; x_n < x_n^*, \quad \lim_{n\to\infty} x_n = 0 . \tag{9}$$

Let

$$h(0) = a_0, \quad h(x) = w_{n+1}(x) + \varphi_n(x, x_n) \quad \text{for} \quad x \in \langle x_{n+1}, x_n\rangle .$$

From (9) we deduce that the function $h(x)$ is defined in the interval $\langle 0, x_3\rangle$. We shall show that the function $h(x)$ satisfies the thesis of the theorem.

For $x \in \langle x_{n+1}, x_n\rangle$ from (4) and (9) we deduce

$$w_{n+1}(x) > f(x), \quad \varphi_n(x, x_n) \geqslant 0 ,$$

hence

$$h(x) > f(x) \quad \text{for} \quad x \in \langle x_{n+1}, x_n\rangle .$$

From this it follows that the inequality

$$h(x) > f(x) \tag{10}$$

holds in the interval $\langle 0, x_3\rangle$.

Since for $x \in \langle x_{n+1}, x_n\rangle$

$$\varphi_n(x, x_n) \leqslant w_n(x) - w_{n+1}(x) ,$$

therefore in the interval $\langle x_{n+1}, x_n\rangle$ we have

$$h(x) < w_n(x) \qquad n = 3, 4, \ldots$$

Since, for $0 < x < x_{n+2}$

$$w_{n+1}(x) > w_{n+2}(x) > w_{n+3}(x)$$

therefore by induction we have

$$w_{n+1}(x) > h(x) \quad \text{for} \quad 0 < x < x_{n+2} \qquad n = 1, 2, \ldots \tag{11}$$

From (10) and (11) we have for $0 < x < x_{n+2}$

$$\frac{f(x) - p_{n-1}(x)}{x^n} < \frac{h(x) - p_{n-1}(x)}{x^n} < \frac{w_{n+1}(x) - p_{n-1}(x)}{x^n} \qquad n = 1, 2, \ldots$$

and because

$$\lim_{x\to 0} \frac{w_{n+1}(x) - p_{n-1}(x)}{x^n} = \lim_{x\to 0} \big(a_n + (a_{n+1} + 1)x\big) = a_n ,$$

hence

$$\lim_{x\to 0} \frac{h(x) - p_{n-1}(x)}{x^n} = a_n . \tag{12}$$

From the inequality $f(x) < h(x) < w_3(x)$ $(0 \leqslant x \leqslant x_3)$ and from $\lim\limits_{x\to 0} w_3(x) = a_0$ it follows that

$$\lim_{x\to 0} h(x) = a_0 . \tag{13}$$

From (12) and (13) it follows that $h(x)$ has at the point 0 the asymptotic expansion (1). Since the values of $h(x)$ on the common ends of the intervals $\langle x_{n+1}, x_n\rangle$ and $\langle x_n, x_{n-1}\rangle$, $n = 4, 5, \ldots$, are equal, $h(x)$ is continuous in $\langle 0, x_3\rangle$. The function $h(x)$ belongs to the class C^1 in $\langle x_{n+1}, x_n\rangle$ and since

$$h'_+(x_n) = w_n(x_n) - h'_-(x_n) = w'_{n+1}(x_n) + w'_n(x_n) - w_{n+1}(x_n) = w_n(x_n) \qquad n = 4, 5, \ldots$$

$h(x)$ is a function of class C^1 in the interval $(0, x_3\rangle$. We have

$$h'(0) = \lim_{x\to 0} \frac{h(x) - h(0)}{x} = \lim_{x\to 0} \frac{h(x) - a_0}{x} = a_1 .$$

In the interval $\langle x_{n+1}, x_n\rangle$ from (5), (7) and (8) we have

$$0 \leqslant \varphi'_n(x, x_n) \leqslant w'_n(x_n) - w'_{n+1}(x_n),$$

$$a_1 - \frac{1}{2n+2} < w'_{n+1}(x) + \varphi'_n(x, x_n) < a_1 + \frac{1}{n+1},$$

$$a_1 - \frac{1}{2n+2} < h'(x) < a_1 + \frac{1}{n+1} .$$

Likewise, for $x \in \langle x_{n+k+1}, x_{n+k}\rangle$ we have

$$a_1 - \frac{1}{2(n+k)+2} < h'(x) < a_1 + \frac{1}{n+k+1},$$

therefore

$$a_1 - \frac{1}{2n+2} < h'(x) < a_1 + \frac{1}{n+1} .$$

This proves the continuity of $h(x)$ at the point 0.

The function $-f(x)$ has the asymptotic expansion

$$\text{(1*)} \qquad \sum_{i=1}^{\infty} (-a_i) x^i .$$

Hence there exist a function $\bar{h}(x)$ and a number $\bar{x}_3$ such that $\bar{h}(x) > -f(x)$ in $\langle 0, \bar{x}_3\rangle$, $h(x)$ has the asymptotic expansion (1*) and $h(x)$ is of class C^1.

Let us put $g(x) = -h(x)$, $a = \min(x_3, \bar{x}_3)$. It is easily to see that the function $g(x)$ satisfies the thesis of the theorem.