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On a system of differential equations

1. In the papers [1] and [2] Z. Butlewski considers a system of second order linear differential equations

$$x'' = -A(t)y\,, \qquad y'' = A(t)x\,, \tag{0}$$

where $A(t)$ is continuous and positive for $t \geqslant 0$. Upon passing to polar coordinates $r(t)$, $\varphi(t)$, where

$$r(t) = \sqrt{x^2(t)+y^2(t)} \quad \text{and} \quad x(t) = r(t)\cos\varphi(t)\,, \quad y(t) = r(t)\sin\varphi(t)\,,$$

(0) yields

$$r'' = r\varphi'^2\,, \qquad \frac{d}{dt}(r^2\varphi') = A(t)r^2\,. \tag{1}$$

For this system Z. Butlewski proves the following theorem:

Assume that $A(t) > 0$, $A'(t) > 0$ (*or* $A'(t) < 0$) *for* $t \geqslant 0$ *and* $\lim\limits_{t\to\infty} A(t) = a > 0$. *Moreover the following initial conditions are satisfied*: $r(0) > 0$, $r'(0) \geqslant 0$, $\varphi'(0) \geqslant 0$. *Then there exist the limits of the functions* r'/r *and* φ' *for* $t\to\infty$ *and* $\lim\limits_{t\to\infty} r'(t)/r(t) = \lim\limits_{t\to\infty}\varphi'(t) = \sqrt{a/2}$.

We shall state a more general theorem. To this purpose we shall consider the following system of non-linear equations

$$u' = v^2 - u^2\,, \qquad v' = A(t) - 2uv \tag{2}$$

and we shall prove the following:

Theorem 1. *Assume that* $A(t)$ *is continuous, positive for* $t \geqslant 0$ *and* $\lim\limits_{t\to\infty} A(t) = a > 0$. *Then all trajectories* $u = u(t)$, $v = v(t)$ $\big(u(0) > -v(0)\big)$ *of the system* (2) *tend to the point* $u = \sqrt{a/2}$, $v = \sqrt{a/2}$ *when* $t \to \infty$.

From the theorem 1 it will follow, as simple conclusion, the following generalization of theorem of Butlewski:

About the system (1) *we assume that* $A(t)$ *is continuous, positive for* $t \geqslant 0$ *and* $\lim_{t\to\infty} A(t) = a > 0$. *Then for every trajectory* $r = r(t)$, $\varphi = \varphi(t)$ *of this system such that* $r(0) > 0$, $r'(0) \geqslant 0$, $r'(0)/r(0) + \varphi'(0) > 0$ *we have*

$$\lim_{t\to\infty} \frac{r'(t)}{r(t)} = \lim_{t\to\infty} \varphi'(t) = \sqrt{\frac{a}{2}}.$$

Indeed, from these assumptions it follows that $r(t) > 0$ for $t \geqslant 0$ ([1], p. 97). Setting $u(t) = r'(t)/r(t)$, $v(t) = \varphi'(t)$, (1) yields (2) and moreover all assumptions of the theorem 1 are satisfied. For this reason it remains to prove the theorem 1 only.

2. We shall first consider the reduced autonomous system

$$u' = v^2 - u^2, \quad v' = a - 2uv \tag{3}$$

obtained from (2) by substitution of the function $A(t)$ by its limit value a. This system has two critical points: $A(\sqrt{a/2}, \sqrt{a/2})$ and $B(-\sqrt{a/2}, -\sqrt{a/2})$. We study the character of the first of them. The transformation of coordinates

$$u = \bar{u} + \sqrt{\frac{a}{2}}, \quad v = \bar{v} + \sqrt{\frac{a}{2}}$$

carries the critical point A to the origin and reduces the system (3) to the following one

$$u' = -2\sqrt{\frac{a}{2}}u + 2\sqrt{\frac{a}{2}}v - u^2 + v^2, \quad v' = -2\sqrt{\frac{a}{2}}u - 2\sqrt{\frac{a}{2}}v - 2uv. \tag{3'}$$

For simplification of notation we use again the previous variables u, v instead of $\bar{u}, \bar{v}$. The first approximation of the system (3′) is the linear system

$$u' = -2\sqrt{\frac{a}{2}}u + 2\sqrt{\frac{a}{2}}v, \quad v' = -2\sqrt{\frac{a}{2}}u - 2\sqrt{\frac{a}{2}}v. \tag{3''}$$

The characteristic equation for (3″) is

$$\lambda^2 + 4\sqrt{\frac{a}{2}}\lambda + 4a = 0.$$

The characteristic roots

$$\lambda_1 = -2\sqrt{\frac{a}{2}} - 2i\sqrt{\frac{a}{2}}, \quad \lambda_2 = -2\sqrt{\frac{a}{2}} + 2i\sqrt{\frac{a}{2}}$$

are complex numbers with negative real parts. Therefore the point $(0, 0)$ is local stable focus for the first approximation (3″) and also for system (3′) ([3], ch. IX). It follows that the critical point A is local stable focus for the system (3). In the same manner we may study the character of the second critical point. This shows that the point B is local unstable focus for the system (3). Thus we get the following property:

(a) *All trajectories of the system* (3), *which are going out from sufficiently small vicinity of critical point A, are tending to A as $t \to \infty$.*

(b) *All trajectories of the system* (3), *which are going out from sufficiently small vicinity of critical point B, are tending to B as $t \to -\infty$.*

We shall prove the following stronger property:

(a′) *All trajectories of the system* (3), *which are going out from the half-plane $u > -v$, are tending to the point A as $t \to \infty$.*

(b′) *All trajectories of the system* (3), *which are going out from the half-plane $u < -v$, are tending to the point B as $t \to -\infty$.*

Proof of (a′). It suffices to find a family of closed curves

$$V_c: \quad V(u, v) = c, \quad c \in \langle \alpha, \beta)$$

such that:

1° the function $V(u, v)$ is defined and continuous in the half-plane $u > -v$,

2° if $c < d$ then V_c is contained in the interior of V_d,

3° the curves V_c, $\alpha \leqslant c < \beta$, cover the whole half-plane $u > -v$,

4° the curve V_α reduces to the point A,

5° for every trajectory $u = u(t)$, $v = v(t)$, $u(0) > -v(0)$, of the system (3) the function

$$c(t) = V\big(u(t), v(t)\big)$$

tends to α when $t \to \infty$.

It is easy to show that such family of curves for the system (3) is, for example, the following family of circles (Fig. 1):

$$\frac{v^2 + u^2 + a}{2(u+v)} = c, \qquad c \in \left\langle \sqrt{\frac{a}{2}}, +\infty \right).$$

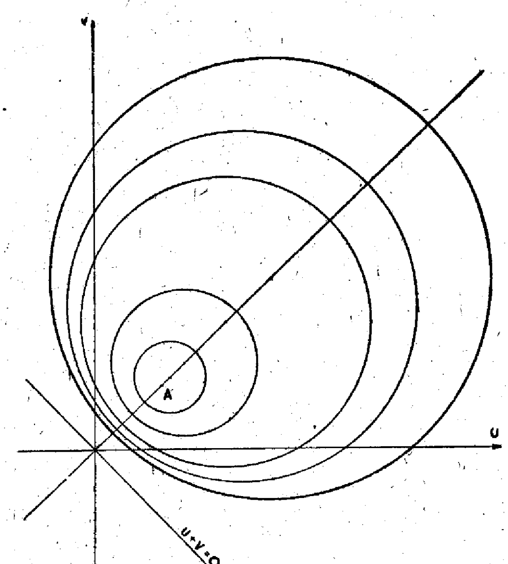

Fig. 1