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Integral formulas for functions holomorphic in convex n-circular domains

1. A domain D in the space C^n of n complex variables $z = (z_1, \ldots, z_n)$ $n \geqslant 1$, is called *a complete n-circular domain* (with centre at 0), if along with arbitrary its point $z^0 = (z_1^0, \ldots, z_n^0)$ D contains the closed polycylinder $\{z\colon |z_k| \leqslant |z_k^0|, k = 1, \ldots, n\}$.

Let Δ be the $(n-1)$-dimensional simplex

$$\Delta = \{\tau = (\tau_1, \ldots, \tau_n)\colon \tau_1 + \ldots + \tau_n = 1, \tau_1 > 0, \ldots, \tau_n > 0\}. \tag{1}$$

We shall say that a bounded domain D in C^n is *of the type* (T) (or simply $D \in (\mathrm{T})$), if there exist positive real functions

$$r_k = r_k(\tau) \qquad k = 1, \ldots, n \tag{2}$$

defined and continuous for $\tau \in \Delta$ such that

$$D = \bigcup_{\tau \in \Delta} \{z\colon |z_k| < r_k(\tau), \ k = 1, \ldots, n\} \tag{3}$$

and

$$D = \text{interior of} \bigcap_{\tau \in \Delta} \left\{z\colon \sum_{k=1}^{n} \frac{\tau_k}{r_k(\tau)} |z_k| < 1\right\}. \tag{4}$$

If $D \in (\mathrm{T})$, then by (3) D is a complete n-circular domain. Moreover the set $S = \{z\colon z_k = r_k(\tau)\exp i\theta_k, 0 \leqslant \theta_k \leqslant 2\pi, k = 1, \ldots, n\}$ belongs to the boundary ∂D of D. There are domains $D \in (\mathrm{T})$ for which S is smaller than ∂D (compare examples 3° and 4° below).

Observe that

$$\Delta = \{\tau\colon \tau_1 = 1 - \tau_2 - \ldots - \tau_n, \ (\tau_2, \ldots, \tau_n) \in \Delta^*\}, \tag{5}$$

where

$$\Delta^* = \{(\tau_2, \ldots, \tau_n)\colon 0 < \tau_2 < 1, 0 < \tau_3 < 1 - \tau_2, \ldots, 0 < \tau_n < 1 - \tau_2 - \ldots - \tau_{n-1}\}. \tag{6}$$

Given any function f holomorphic in a domain $D \in (\mathrm{T})$ continuous in $D \subset S$ along with its partial derivates of degree $\leqslant \mu$, put

$$F_0(\zeta, r, \theta) = f(\zeta r_1, \zeta r_2 e^{i\theta_2}, \dots, \zeta r_n e^{i\theta_n}), \tag{7}$$

$$F_{k+1} = (n-k)F_k + \zeta \frac{\partial}{\partial \zeta} F_k, \quad k = 0, \dots, n-2 \tag{8}$$

where $|\zeta| \leqslant 1$, $0 \leqslant \theta_k \leqslant 2\pi$ $(k = 2, \dots, n)$ and $r_k = r_k(\tau)$ are functions (2).

The purpose of this paper is to prove the following theorems.

Theorem 1. *If $D \in (\mathrm{T})$ and if a holomorphic function f and all its partial derivatives of degree $\leqslant \mu$, where $0 \leqslant \mu \leqslant n-1$, are continuous in $D \cup S$, then for $k = 0, \dots, \mu$ and $z \in D$*

$$f(z) = \frac{(n-k-1)!}{(2\pi)^n i} \int d\omega_\tau \int d\omega_\theta \frac{\zeta^{n-k-1} F_k(\zeta, r, \theta)}{(\zeta - u)^{n-k}} d\zeta, \tag{9}$$

where

$$u = \frac{\tau_1}{r_1(\tau)} z_1 + \frac{\tau_2}{r_2(\tau)} z_2 e^{-i\theta_2} + \dots + \frac{\tau_n}{r_n(\tau)} z_n e^{-i\theta_n}, \tag{10}$$

$$\int d\omega_\tau = \int_{\Delta^*} d\tau_2 \dots d\tau_n, \quad \int d\omega_\theta = \int_0^{2\pi} d\theta_2 \int_0^{2\pi} d\theta_2 \dots \int_0^{2\pi} d\theta_n$$

and the circle $|\zeta| = 1$ is oriented positively with respect to its interior.

Theorem 2. *The necessary and sufficient condition that a bounded domain $D \subset C^n$ belong to (T) is that D be a convex complete n-circular domain.*

Given any bounded convex complete n-circular domain D there is exactly one system of functions (2) satisfying (3) and (4).

Remarks. 1. If $n = 2$, the integral formulas (9) are identical with integral formulas of Temlakov, who derived them for a subset of domains $D \in (\mathrm{T})$ ([2], p. 346—351).

2. For a subset of domains $D \in (\mathrm{T})$ in C^n, $n \geqslant 3$, different but equivalent to (9) integral formulas has been proved by Aizenberg and Li Tche Gon ([2], p. 346—351).

3. If $n = 2$ and the boundary ∂D of D besides convexity satisfies some additional regularity conditions, Theorem 2 has been first proved by Temlakov ([2], p. 346—351).

2. Examples. 1° Let $D = \{z: \sum_{k=1}^{n} |z_k|^2 < \varrho^2\}$. Then $r_k(\tau) = \sqrt{\tau_k}$, $k = 1, \dots, n$.

2° Let $D = \{z: \sum_{k=1}^{n} p_k |z_k| < 1\}$ $(p_k > 0,\ k = 1, \dots, n)$. Then $r_k(\tau) = p_k^{-1} \tau_k$, $k = 1, \dots, n$.

3° Let $D = \{z: |z_k| < \varrho_k, k = 1, \dots, n\}$ $(\varrho_k > 0, k = 1, \dots, n)$. Then $r_k(\tau) = \varrho_k$, $k = 1, \dots, n$.

4° Let $n = 2$ and $D = \{(z_1, z_2): p_1|z_1| + p_2|z_2| < 1\} \cap \{(z_1, z_2): q_1|z_1| + q_2|z_2| < 1\}$, where $0 < p_1 < q_1 < \infty$, $0 < q_2 < p_2 < \infty$ (Fig. 1). Suppose $\xi, \eta > 0$ and $p_1\xi +$

$+p_2\eta = q_1\xi + q_2\eta = 1$. Then

$$r_1(\tau) = \begin{cases} \tau_1/p_1 & \text{if} \quad 0 < \tau_1 < \xi p_1 \\ \xi & \text{if} \quad \xi p_1 \leqslant \tau_1 \leqslant \xi q_1 \\ \tau_1/q_1 & \text{if} \quad \xi q_1 < \tau_1 < 1 \end{cases}$$

$$r_2(\tau) = \begin{cases} \tau_2/p_2 & \text{if} \quad 0 < 1-\tau_2 < \xi p_1 \\ \eta & \text{if} \quad \xi p_1 \leqslant 1-\tau_2 \leqslant q_1 \\ \tau_2/q_2 & \text{if} \quad \xi q_1 < 1-\tau_2 < 1 . \end{cases}$$

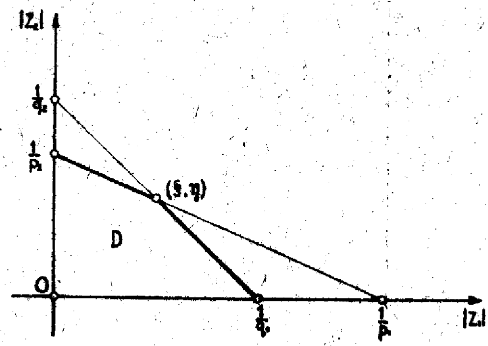

Fig. 1

5° Let $n=2$ and $D = [\{(z_1, z_2) : |z_1| < a_1, |z_2| < b_1\} \cup \{(z_1, z_2) : |z_1| < a_2, |z_2| < b_2\}] \cap \{(z_1, z_2) : |z_1|/a_3 + |z_2|/b_3 < 1\}$, where $0 < a_1 < a_2 < a_3$, $0 < b_2 < b_1 < b_3$ (Fig. 2). Here the functions $r_1(\tau)$ and $r_2(\tau)$ are given by

$$r_1(\tau) = \begin{cases} a_1 & \text{if} \quad 0 < \tau_1 < a_1/a_3 \\ a_3\tau_1 & \text{if} \quad a_1/a_3 \leqslant \tau_1 \leqslant a_2/a_3 \\ a_2 & \text{if} \quad a_2/a_3 < \tau_1 < 1 \end{cases}$$

$$r_2(\tau) = \begin{cases} b_1 & \text{if} \quad 0 < 1-\tau_2 < a_1/a_3 \\ b_3\tau_2 & \text{if} \quad a_1/a_3 \leqslant 1-\tau_2 \leqslant a_2/a_3 \\ b_2 & \text{if} \quad a_2/a_3 < 1-\tau_2 < 1 . \end{cases}$$

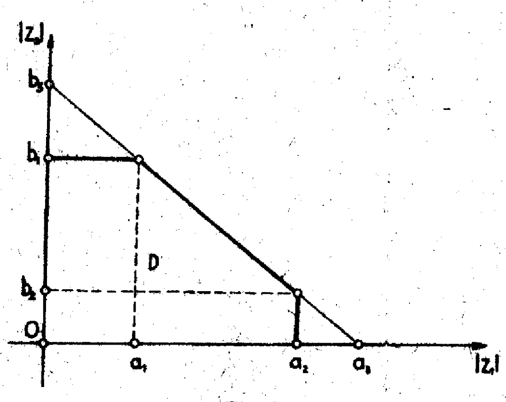

Fig. 2

3. **Proof of Theorem 1.** First of all observe that the right-hand side $R(z)$ of (9) is well defined for $z \in D$ and moreover the function $R(z)$ is holomorphic in D. Next observe that due to Cauchy integral formula we have

$$I = \frac{(n-k-1)!}{2\pi i} \int\limits_{|\zeta|=1} \frac{\zeta^{n-k-1} F_k(\zeta, r, \theta)}{(\zeta-u)^{n-k}} d\zeta = \frac{\partial^{n-k-1}}{\partial u^{n-k-1}} [u^{n-k-1} F_k(u, r, \theta)],$$

whence in virtue of (8)

$$I = \frac{\partial^{n-k-1}}{\partial u^{n-k-1}} \left\{ u^{n-k-1} \left((n-k-1) F_{k-1} + u \frac{\partial}{\partial u} F_{k-1} \right) \right\}$$
$$= \frac{\partial^{n-k}}{\partial u^{n-k}} [u^{n-k} F_{k-1}] = \ldots = \frac{\partial^{n-1}}{\partial u^{n-1}} [u^{n-1} F_0].$$

This implies that the right-hand side $R(z)$ of (9) is the same for all $k = 0, \ldots, \mu$, namely

$$R(z) = \frac{1}{(2\pi)^{n-1}} \int d\omega_\tau \frac{\partial^{n-1}}{\partial u^{n-1}} [u^{n-1} F_0(u, r, \theta)] d\omega_\theta . \tag{11}$$

If $\|z\| \leqslant \varrho$, where $\varrho > 0$ is small enough, the value of the right-handside of (9) does not change when we replace the road of integration $|\zeta| = 1$ by the circle $|\zeta| = \frac{1}{2}$. However the closure of the set $\{z\colon z = \big(r_1(\tau)\zeta, r_2(\tau)\zeta e^{i\theta_2}, \ldots \ldots, r_n(\tau)\zeta e^{i\theta_n}\big)$, where $|\zeta| = \frac{1}{2}$, $\tau \in \Delta$ and $0 \leqslant \theta_k \leqslant 2\pi$, $k = 1, \ldots, n\}$ belongs to D. Since every function $f(z)$ holomorphic in D may be approximated by polynomials in $z_1, \ldots, z_n$ uniformly on every compact subset of D, therefore in virtue of additiveness property of integrals, this implies that it is enough to prove the formulas (9) only for monomials. Namely, after having proved (9) for monomials (thus for polynomials), we shall get by the approximation procedure $R(z) = f(z)$ for $\|z\| \leqslant \varrho$. But $f(z)$ and $R(z)$ being both holomorphic in D, this implies that $f(z) = R(z)$ for $z \in D$.

Let $f(z)$ be an arbitrary monomial $f(z) = z_1^{\mu_1} \ldots z_n^{\mu_n}$, $\mu_1 + \ldots + \mu_n = m$, $m = 0, 1, \ldots$ In this case $u^{n-1} F_0(u, r, \theta) = u^{m+n-1} r_1^{\mu_1} \ldots r_n^{\mu_n} \exp(i\mu_2\theta_2 + \ldots + i\mu_n\theta_n)$. Therefore due to (11)

$$R(z) = \frac{1}{(2\pi)^{n-1}} (m+n-1) \ldots (m+1) \int r_1^{\mu_1} \ldots r_n^{\mu_n} d\omega_\tau \int u^m e^{i(\mu_2\theta_2 + \ldots + \mu_n\theta_n)} d\omega_\theta .$$

One easily checks that

$$\frac{1}{(2\pi)^{n-1}} \int u^m e^{i(\mu_2\theta_2 + \ldots + \mu_n\theta_n)} d\omega_\theta = \binom{m}{\mu_n} \binom{m-\mu_n}{\mu_{n-1}} \ldots \binom{\mu_1+\mu_2}{\mu_2} \cdot \left(\frac{\tau_1 z_1}{r_1}\right)^{\mu_1} \ldots \left(\frac{\tau_n z_n}{r_n}\right)^{\mu_n} .$$

So

$$R(z) = \frac{(m+n-1)!}{\mu_1! \ldots \mu_n!} z_1^{\mu_1} \ldots z_n^{\mu_n} \int \tau_2^{\mu_2} \ldots \tau_n^{\mu_n} (1 - \tau_2 - \ldots - \tau_n)^{\mu_1} d\omega_\tau$$

whence $R(z) = z_1^{\mu_1} \dots z_n^{\mu_n}$, because by (6)

$$\int \tau_2^{\mu_2} \dots \tau_n^{\mu_n} (1-\tau_2-\dots-\tau_n)^{\mu_1} d\omega_\tau =$$

$$= \int_0^1 \tau_2^{\mu_2} d\tau_2 \int_0^{1-\tau_2} \tau_0^{\mu_3} d\tau_3 \dots \int_0^{1-\tau_2-\dots-\tau_{n-1}} \tau_n^{\mu_n} (1-\tau_2-\dots-\tau_n)^{\mu_1} d\tau_n = \frac{\mu_1! \dots \mu_n!}{(m+n-1)!}.$$

To compute this integral observe that (after repetition suitably many times the integration by parts)

$$\int_0^{1-\tau_2-\dots-\tau_{k-1}} \tau_k^{\mu} (1-\tau_2-\dots-\tau_k)^{\nu} d\tau_k = \frac{\mu!\,\nu!}{(\mu+\nu+1)!} (1-\tau_2-\dots-\tau_k)^{\mu+\nu+1}.$$

Thus the proof of Theorem 1 is completed.

4. **Proof of Theorem 2.** Sufficiency follows directly from the definition of (T). Before we proceed to the proof of the necessity let us make few introductory observations. If D is a bounded convex complete n-circular domain, then to every boundary point $z^0 = (z_1^0, \dots, z_n^0) \in \partial D$ there is a *support hypersurface* H

$$H = \Big\{z: \sum_{k=1}^{n} p_k |z_k| = \sum_{k=1}^{n} p_k |z_k^0| = 1\Big\} \tag{12}$$

where $p_k = p_k(z^0) = p_k(|z_1^0|, \dots, |z_n^0|)$, $k = 1, \dots, n$ are nonnegative real numbers and D is contained in $\{z: \sum_{k=1}^{n} p_k |z_k| < 1\}$. The support hypersurfaces H of D are in one-to-one correspondence with the support hyperplanes of D_+, where D_+ is a map of D under the standard transformation of n-circular domain into n-dimensional real space. All our further considerations might be carried over for domains D_+ instead for D.

Given a bounded convex complete n-circular domain D with center at 0, denote by P the set of all $p = (p_1, \dots, p_n)$ such that $p_k > 0$, $k = 1, \dots, n$, and $\{z: \sum_{k=1}^{n} p_k |z_k| = 1\}$ is a support hypersurface of D at a point $z^0 = (z_1^0, \dots, z_n^0) \in \partial D$ such that $z_1^0 z_2^0 \dots z_n^0 \neq 0$. One checks that

$$D = \text{interior of} \bigcap_{p \in P} \Big\{z: \sum_{k=1}^{n} p_k |z_k| < 1\Big\}. \tag{13}$$

This implies that to find functions (2) satisfying (3) and (4) means to find a special parametrization of the support hypersurfaces (12) of D.

It is easily to check that domains D_ν, $\nu = 1, 2, \dots$, defined by

$$D_\nu = \text{interior of} \bigcap_{p \in P} \Bigg\{z: \sum_{k=1}^{n} \frac{p_k}{1 + \frac{1}{\nu}\|p\|} |z_k| < 1\Bigg\}, \tag{14}$$