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# On some sequences connected with the extremal points

Let $R$ be an arbitrary metric space and $E$ a set in $R$ being the sum of two disjoined compact sets $E_1$ and $E_2$. Let $\omega(p, q)$ be a real-valued continuous function of two points $p$ and $q$ defined in $R$. Suppose that $\omega(p, q) = \omega(q, p) \geqslant 0$.

Let $q^{(n)} = (q_0^{(n)}, \ldots, q_n^{(n)})$ (denoted also by $(q_0, \ldots, q_n)$) be an extremal system of $E$ (see [4], p. 260) and suppose that $n > 3$, $\nu > 1$, $n-\nu > 1$ and $q_0, \ldots, q_\nu \in E_1$, $q_{\nu+1}, \ldots, q_n \in E_2$. Consider three sequences

$$V_n^1 = \Big[\prod_{0\leqslant i<k\leqslant \nu} \omega(q_i, q_k)\Big]^{2/\nu(\nu+1)}, \qquad V_n^2 = \Big[\prod_{\nu+1\leqslant i<k\leqslant n} \omega(q_i, q_k)\Big]^{2/(n-\nu)(n-\nu-1)}$$

$$V_n^3 = \Big[\prod_{i=0}^{\nu} \prod_{k=\nu+1}^{n} \omega(q_i, q_k)\Big]^{1/(\nu+1)(n-\nu)}.$$

The object of the present paper is to prove the following

Theorem. *If the extremal measure $\eta$ i.e. the measure realising*

$$\inf_{\mu\geqslant 0} \int_E\int_E \log\frac{1}{\omega(p, q)}\, d\mu(p)\, d\mu(q) \qquad \big(\mu(E) = 1\,,\ \mu(E\cdot e) = 0 \quad \text{if} \quad E\cdot e = 0\big)$$

*is unique and $\eta(E_1) > 0$, $\eta(E_2) > 0$, then the sequences $V_n^1, V_n^2, V_n^3$ are convergent.*

Proof. Let $\delta$ be an arbitrary positive number. Denote by $\omega_\delta(p, q)$ the function defined by

$$\omega_\delta(p, q) = \begin{cases} \omega(p, q) & \text{for} \quad \omega(p, q) \geqslant \delta \\ \delta & \text{for} \quad \omega(p, q) < \delta \end{cases}$$

and consider the sequences

$$V_n^{1,\delta} = \Big[\prod_{0\leqslant i<k\leqslant \nu} \omega_\delta(q_i, q_k)\Big]^{2/\nu(\nu+1)}, \qquad V_n^{2,\delta} = \Big[\prod_{\nu+1\leqslant i<k\leqslant n} \omega_\delta(q_i, q_k)\Big]^{2/(n-\nu)(n-\nu-1)}$$

$$V_n^{3,\delta} = \Big[\prod_{i=0}^{\nu} \prod_{k=\nu+1}^{n} \omega_\delta(q_i, q_k)\Big]^{1/(\nu+1)(n-\nu)}.$$

To each system of extremal points $q^{(n)}$ there corresponds a function of the Borel set $e$ defined by

$$\mu_n(e) = \begin{cases} 0\,, & \text{if } e \text{ does not contain any point of } q^{(n)} \\ \dfrac{k}{n+1}\,, & \text{if } e \text{ contains } k \text{ points of } q^{(n)}. \end{cases}$$

Since the extremal measure $\eta$ is unique, so (see [2])

$$\mu_n \to \eta \tag{1}$$

and so $\nu \to \infty$, $n-\nu \to \infty$ when $n \to \infty$. Since $\log \omega_\delta(p, q)$ is continuous, so $\infty > \sup_{p,q \in E} \log \omega_\delta(p, q) \geqslant \log \omega_\delta(p, q) \geqslant \log \delta$. Therefore due to (1)

$$V_n^{1,\delta} = \exp \frac{2}{\nu(\nu+1)} \sum_{0 \leqslant i < k \leqslant \nu} \log \omega_\delta(q_i, q_k) = \exp \frac{1}{\nu(\nu+1)} \sum_{0 \leqslant i,k \leqslant \nu} \log \omega_\delta(q_i, q_k) =$$

$$= \exp \frac{1}{\nu(\nu+1)} \Big( \sum_{0 \leqslant i,k \leqslant \nu} \log \omega_\delta(q_i, q_k) - \sum_{i=0}^{\nu} \log \omega_\delta(q_i, q_i) \Big) =$$

$$= \exp \Big( \frac{(n+1)^2}{\nu(\nu+1)} \int_{E_1} \int_{E_1} \log \omega_\delta(p, q)\, d\mu_n(p)\, d\mu_n(q) - \frac{1}{\nu(\nu+1)} \sum_{i=0}^{\nu} \log \omega_\delta(q_i, q_i) \to$$

$$\to \exp \frac{1}{[\eta(E_1)]^2} \int_{E_1} \int_{E_1} \log \omega_\delta(p, q)\, d\eta(p)\, d\eta(q), \quad \text{if} \quad n \to \infty.$$

By the similar way we can prove that if $n \to \infty$ then

$$V_n^{2,\delta} \to \exp \frac{1}{[\eta(E_2)]^2} \int_{E_2} \int_{E_2} \log \omega_\delta(p, q)\, d\eta(p)\, d\eta(q)$$

and

$$V_n^{3,\delta} \to \exp \frac{1}{\eta(E_1)\eta(E_2)} \int_{E_1} \int_{E_2} \log \omega_\delta(p, q)\, d\eta(p)\, d\eta(q).$$

Hence

$$(V_n^{1,\delta})^{\frac{\nu}{n} \cdot \frac{\nu+1}{n+1}} \to \exp \int_{E_1} \int_{E_1} \log \omega_\delta(p, q)\, d\eta(p)\, d\eta(q)$$

$$(V_n^{2,\delta})^{\frac{n-\nu}{n} \cdot \frac{n-\nu-1}{n+1}} \to \exp \int_{E_2} \int_{E_2} \log \omega_\delta(p, q)\, d\eta(p)\, d\eta(q) \tag{2}$$

$$(V_n^{3,\delta})^{2 \cdot \frac{n-\nu}{n} \cdot \frac{\nu+1}{n+1}} \to \exp 2 \int_{E_1} \int_{E_2} \log \omega_\delta(p, q)\, d\eta(p)\, d\eta(q).$$

Since $V_n^{i,\delta} \geqslant V_n^i$, $i = 1, 2, 3$ and (see [2])

$$(V_n^1)^{\frac{\nu}{n} \cdot \frac{\nu+1}{n+1}} \cdot (V_n^2)^{\frac{n-\nu}{n} \cdot \frac{n-\nu-1}{n+1}} \cdot (V_n^3)^{2 \frac{n-\nu}{n} \cdot \frac{\nu+1}{n+1}} =$$

$$= \Big[ \prod_{0 \leqslant i < k \leqslant n} \omega(q_i, q_k) \Big]^{2/n(n+1)} \to \exp \int_E \int_E \log \omega(p, q)\, d\eta(p)\, d\eta(q)$$

and since

$$\int_E\int_E \log\omega_\delta(p,q)\,d\eta(p)\,d\eta(q)\to\int_E\int_E \log\omega(p,q)\,d\eta(p)\,d\eta(q)$$

if $\delta\to 0$, so by (2) the sequences

$$\{(V_n^1)^{\frac{\nu}{n}\cdot\frac{\nu+1}{n+1}}\},\quad \{(V_n^2)^{\frac{n-\nu}{n}\cdot\frac{n-\nu-1}{n+1}}\},\quad \{(V_n^3)^{\frac{\nu}{n}\cdot\frac{n-\nu}{n+1}}\}$$

are convergent respectively to

$$\exp\int_{E_1}\int_{E_1}\log\omega(p,q)\,d\eta(p)\,d\eta(q),\qquad \exp\int_{E_2}\int_{E_2}\log\omega(p,q)\,d\eta(p)\,d\eta(q),$$

$$\exp\int_{E_1}\int_{E_2}\log\omega(p,q)\,d\eta(p)\,d\eta(q).$$

Hence and by (1) it follows that the sequences $V_n^1, V_n^2, V_n^3$ are convergent.

**Remark.** Let $R$ be a complex plane and $\omega(z,\zeta)=|z-\zeta|$. The complementary set $R\setminus E$ consists of at most enumerably infinite number of domains. One of them contains $z=\infty$. Let $F$ be its boundary. If

$$\text{(3)}\qquad \inf_{\mu\geqslant 0}\int_{E_iF}\int_{E_iF}\log\frac{1}{|z-\zeta|}\,d\mu(z)\,d\mu(\zeta)<\infty,\qquad i=1,2$$

then (see [1] and [3]) the measure $\eta$ is unique and $\eta(E_1)>0$, $\eta(E_2)>0$. Hence and from our theorem it follows that if the conditions (3) are satisfied then the answer to the problem of F. Leja (see Colloquium Mathematicum 7 (1959), p. 152) is positive. If the conditions (3) are not satisfied the answer may be negative what follows from the following example.

Let $E_1$ consists of the points $z=0$, $z=1$, $z=2$ and $E_2$ of the points $z=-1$, $z=-2$. Let $q^{(2)}=(2,-2)$, $q^{(3)}=(2,0,-2)$, $q^{(4)}=(2,1,-1,-2)$, $q^{(5)}=(2,1,0,-1,-2)$ and for $n\geqslant 3$ $q^{(2n)}=(2,1,0,-1,-1,\ldots,-1)$, $q^{(2n+1)}=(2,1,-1,-1,\ldots,-1)$. It is easy to see that $q^{(n)}$ defined in this way is the $n$-th extremal system of $E$ and for $n\geqslant 3$ $V_{2n}^1=\sqrt[3]{2}$ and $V_{2n+1}^1=1$ so the sequence $\{V_n^1\}$ is not convergent.

## REFERENCES


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[4] F. Leja, *Teoria funkcji analitycznych*, Warszawa 1957.