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On some inequalities

1. Definition 1: We call a set P *partly ordered* [1], [2] if for some pairs of elements $x, y \epsilon P$, a relation $x \leqslant y$ is defined, in such a way that: (a) for each $x \epsilon P$, $x \leqslant x$, (b) if $x \leqslant y$ and $y \leqslant x$, then $x = y$, (c) if $x \leqslant y$ and $y \leqslant z$, then $x \leqslant y$.

Definition 2. Let P be a partly ordered set and let $Q \subset P$. We call z *upper bound of* Q *in* P if $z \epsilon P$ and if $x \epsilon Q$, then $x \leqslant \hat{z}$.

Definition 3. We call $\hat{z}$ *supremum of the set* Q (shortly sup Q) if $\hat{z}$ is an upper bound of Q in P and if x is an upper bound of Q in P, then $x \leqslant \hat{z}$.

Each partly ordered set can have at the most one supremum.

Definition 4. The set P will be said to satisfy the condition (Π) if the difference $x - y \epsilon P$ is defined for each $x, y \epsilon P$ in such a way that (d) if $x \leqslant y$, then for each $z \epsilon P$ is $x - z \leqslant y - z$, (e) there exists an element $0 \epsilon P$, such that for each $x \epsilon P$ is $x - 0 = x$, (f) $x = y$ if and only if $x - y = 0$.

Definition 5. The set P will be said to satisfy the condition (Π^*), if for each $x, y \epsilon P$, there exists in P $z = \sup\{x, y\}$.

2. Theorem A. *Let us assume that:*

1. *The set P is not empty, partly ordered and fulfills the conditions (Π) and (Π^*).*
2. *the functions $W(x)$ and $L(x)$ are defined in the set P, and are such that $W(P) \subset P$ and $L(P) \subset P$,*
3. *if $x \leqslant L(x)$, then $x \leqslant 0$,*
4. *if $x \leqslant y$, then $W(x) \leqslant W(y)$,*
5. *if $0 \leqslant W(x) - W(y)$, then $W(x) - W(y) \leqslant L(x - y)$,*
6. *w is a solution of the equation*

$$w = W(w) \tag{1}$$

7. *$v \epsilon P$ is such that:*

$$v \leqslant W(v)\,. \tag{2}$$

Then we have

$$v \leqslant w . \tag{3}$$

Proof. Let z be the supremum of the set $\{w, v\}$. Then

$$w \leqslant z \quad \text{and} \quad v \leqslant z . \tag{4}$$

From (4), the condition (c) and the assumption 4 follows that

$$w = W(w) \leqslant W(z) \quad \text{and} \quad v \leqslant W(v) \leqslant W(z) . \tag{5}$$

From (5) and from definition of z as the $\sup\{w, v\}$ follows that $z \leqslant W(z)$. From the condition (d) we have

$$z - w \leqslant W(z) - w = W(z) - W(w) .$$

Then, from (4) and the assumption 5 follows that

$$z - w \leqslant L(z-w) . \tag{6}$$

In view of the assumption 3, the inequality (6) implies that $z - w = 0$. Hence $z = w$, what means that (3) holds.

Remark. It is easy to see that we can assume that the function $L(x)$ is defined only for $0 \leqslant x$ ($x \in P$). It is also easy to see that under the assumptions of Theorem A the equation (1) can have at most one solution in P.

3. As an example we shall consider the following equation

$$u(X) = f(X) + \int_E F\big(X, Y, u(Y)\big) dY \tag{7}$$

where $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_n\}$ and E is a n-dimensional set, and the inequality

$$v(X) \leqslant f(X) + \int_E F\big(X, Y, v(Y)\big) dY . \tag{8}$$

Theorem B. *Assumptions*:

1. $F(X, Y, z)$ *is defined and continuous in* $\widetilde{E} \times E \times (-\infty, +\infty)$,
2. $f(X)$ *is defined and continuous in* $\widetilde{E}$,
3. *if* $z \leqslant \hat{z}$, *then* $F(X, Y, z) \leqslant F(X, Y, \hat{z})$,
4. $|F(X, Y, z) - F(X, Y, \hat{z})| \leqslant l(X, Y, |z - \hat{z}|)$, *where the function* $l(X, Y, z)$ *is defined in* $\widetilde{E} \times E \times (-\infty, +\infty)$ *and such that, if*

$$w(X) \leqslant \int_E l\big(X, Y, w(Y)\big) dY$$

then

$$w(X) \leqslant 0,$$

5. *$u(X)$ is a solution of the equation* (7) *in* $\widetilde{E}$,
6. *$v(X)$ is a continuous function defined in* $\widetilde{E}$ *and fulfills the inequality* (8).

Then in the set $\widetilde{E}$ *we have*

$$v(X) \leqslant u(X). \tag{14}$$

Proof. It is easy to see that the set P of all continuous functions $w(X)$ defined for $X \in E$ fulfills the assumptions of Theorem A, concerning the set P. Now, for $u \in P$, we define $W(u)$ as the function, which in the point u has as the value the point z defined by the formula

$$z(X) = f(X) + \int_E F\big(X, Y, u(Y)\big) dY.$$

By $L(u)$ we denote the function defined in P, which in the point $u \in P$, has as the values the point w defined by the formula

$$w(X) = \int_E l\big(X, Y, u(Y)\big) dY.$$

It is easy to see that the functions $W(u)$ and $L(u)$ fulfill all assumptions of Theorem A. Hence for each solution w of the equation

$$w = W(w)$$

and each function, which fulfills the inequality

$$v \leqslant W(v)$$

we have $v \leqslant w$. This means that for each solution of the equation (7) and for each function $v(X)$ which fulfills the inequality (8), we have

$$v(X) \leqslant w(X).$$

Remark. In particular we can put $l(X, Y, z) = K \cdot z$. If we assume that the function $F(X, Y, z)$ is bounded and that $K < \mu(E)$, where $\mu(E)$ is the measure of the set E, then all assumptions of theorem A are satisfied.

4. Remark. Let us notice that we can Theorem A apply to the non-linear Volterra's type equation

$$y(x) = f(x) + \int_a^x F\big(x, t, y(t)\big) dt$$

and the inequality

$$v(x) \leqslant f(x) + \int_a^x F(x, t, v(t))\, dt .$$

On that way it is possible to prove some theorems, analogous to the theorem B which will contain some special cases of theorems of T. Satô [3].

REFERENCES

[1] G. Birkhoff, *Lattice theory*, New York 1948.

[2] Л. В. Канторович, Б. З. Вулих, А. Г. Пинскер, *Функциональный анализ в полуупорядоченных пространствах*, Москва-Ленинград 1950.

[3] T. Satô, *Sur l'équation intégrale non linéaire de Volterra*, Compositio Mathematica 11 (1953), p. 271—290.