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On periodic solutions of Hamiltonian system of differential equations on the plane

1. In this paper we give sufficient conditions for the existence of the periodic solutions with the period ω of the system

$$\dot{x} = -H_y(x, y) + p_1(t), \quad \dot{y} = H_x(x, y) + p_2(t) \tag{1}$$

where functions $p_1(t), p_2(t)$ are periodic with the period ω.

This note establishes some generalisations of the results obtained by Z. Opial [2], who has examined the equation $\ddot{x} + g(x) = p(t)$.

The Hamiltonian system

$$\dot{x} = -H_y(x, y), \quad \dot{y} = H_x(x, y) \tag{2}$$

has the first integral $H(x, y)$. A solution of the system (2), which crosses a point (x_0, y_0) of the plane is periodic if and only if the curve given by equation $H(x, y) = H(x_0, y_0)$ is closed.

Let us assume that the system (2) has only periodic solutions, i.e. that the curves $H(x, y) = H(x_0, y_0)$ for arbitrary point (x_0, y_0) are closed and that $\liminf\limits_{x_0^2+y_0^2\to\infty} T(x_0, y_0) \geqslant T_0 > 0$, where $T(x_0, y_0)$ denotes the period of solution of (2) which starts at the point (x_0, y_0).

It will be proved further that there exists a positive number $\omega_0 < T_0$ such that if $\omega < \omega_0$ then the system (1) has a periodic solution with the period ω.

The method of proof is similar to the method applied in [2]. In the last part of the note a theorem which generalizes Theorem 2 of [2] is given.

2. Theorem 1. *Assume that the function $H(x, y)$ is of class C^1 and that the following conditions are satisfied:*

$$H_x(x, y)x > 0 \quad \textit{for} \quad x \neq 0, \; H_y(x, y)y > 0 \quad \textit{for} \quad y \neq 0, \tag{3}$$

$$\liminf_{r\to\infty} (H_x^2 + H_y^2) = \infty \quad (r^2 = x^2 + y^2), \tag{4}$$

$$\limsup_{r\to\infty} \frac{|H_x|}{r} \leqslant A, \quad \limsup_{r\to\infty} \frac{|H_y|}{r} \leqslant B, \tag{5}$$

(6) $p_1(t), p_2(t)$ *are continuous and periodic with the period* ω.

If the period ω satisfies inequality

$$\omega < \omega_0 = \frac{4}{\sqrt{A^2+B^2}} \ln \frac{A+B+\sqrt{A^2+B^2}}{A+B-\sqrt{A^2+B^2}} \tag{7}$$

then the system (1) *possesses at least one periodic solution with the period* ω.

By (3) and (4) a curve $H(x, y) = H(x_0, y_0)$ is closed for each point (x_0, y_0). To prove this it is sufficient to notice that $\lim_{r\to\infty} H(x, y) = \infty$, so the set of points $\{(x, y): H(x, y) = \text{const}\}$ is bounded. Therefore it must form the closed curve.

(3) and (5) imply that $\liminf_{x_0^2+y_0^2\to\infty} T(x_0, y_0) \geqslant T_0 > 0$. Let us take the equation

$$\dot{\varphi} = \frac{H_x(r\cos\varphi_1 r\sin\varphi)}{r}\cos\varphi + \frac{H_y(r\cos\varphi, r\sin\varphi)}{r}\sin\varphi$$

which was obtained from (2) by introduction the polar coordinates (r, φ). For sufficiently large r we have inequality

$$0 < \dot{\varphi} < A_1|\cos\varphi| + B_1|\sin\varphi| \qquad (A_1 = A+\varepsilon,\ B_1 = B+\varepsilon)$$

from which it follows that

$$T(x_0, y_0) > \int_0^{2\pi} \frac{d\varphi}{A_1|\cos\varphi| + B_1|\sin\varphi|} > 0$$

if $x_0^2 + y_0^2$ is large enough.

Before proving Theorem 1 the following lemma will be proved.

Lemma. *Let us denote by $L(t)$ a solution of* (1) *given by the functions* $x = x(t, u, v)$, $y = y(t, u, v)$, *for which* $x(t_0, u, v) = u$, $y(t_0, u, v) = v$.

Assume that for $t = t_1$ the arc $L(t)$ crosses the first time the ray $x = us$, $y = vs$, $(s \geqslant 0)$. *$L_1(t)$ is the subarc of $L(t)$, considered for $t \in \langle t_0, t_1\rangle$. If* (3), (4), (5), (6) *hold, then for arbitrary chosen constants $\varrho > 0$, $\delta > 0$ there exists $r_0 > 0$ such that:*

(A) *the arc $L_1(t)$ lies outside the circle $x^2+y^2 = \varrho^2$,*

(B) $t_1 - t_0 > \omega_0 - \delta$, *if* $u^2+v^2 \geqslant r_0^2$.

Proof of (A). We base on the following simple property of the curve $G(x, y; q, r) = H(x, y) + qx + ry = C$ (C-constant): for $\varrho > 0$ and fixed q, r the distance from the curve $G(x, y, q, r) = C$ to the origin of coordinates is larger than ϱ if the constant C is large enough.

Let, for example, u, v be positive numbers (in other cases the proof is similar to the one given below) and let us construct the "spiral" curve K which begins at the point (u, v). The curve K (see the figure) consists of arcs $K_1, \ldots, K_5$ defined by conditions

$$K_i: G(x, y, q_i, r_i) = C_i \qquad (x, y) \in D_i \qquad (i = 1, \ldots, 5)$$

where $C_1 = G(u, v; q_1, r_1)$ and C_i $(i = 2, \ldots, 5)$ are so chosen that K_i is a prolongation of K_{i-1} $(i = 2, \ldots, 5)$. D_i are the parts of (x, y)-plane as marked on the figure (Fig. 1).

Numbers q_i, r_i satisfy the inequalities

$$\begin{aligned} H_x(x, y)\big(r_i + p_1(t)\big) > 0 \\ H_y(x, y)\big(-q_i + p_2(t)\big) > 0 \end{aligned} \qquad (x, y) \in D_i \quad (i = 1, \ldots, 5)\,.$$

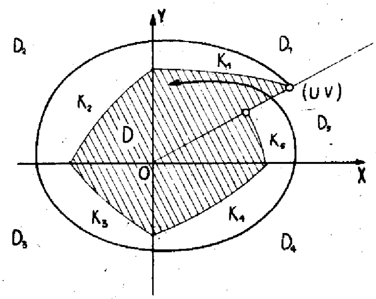

Fig. 1

Since $p_1(t), p_2(t)$ are periodic, they are bounded; let us put

$$p = \max_t \big(|p_1(t)|, |p_2(t)|\big)$$

and

$$M = \max(p\,|q_1| + p\,|r_1|, \ldots, p\,|q_5| + p\,|r_5|)\,, \qquad m = \min_t \big(|r_1 + p_1(t)|, \ldots, |r_5 + p_1(t)|\,, \\ |-q_1 + p_2(t)|, \ldots, |-q_5 + p_2(t)|\big)\,.$$

From (4) it follows that there exists R such that $|H_x| + |H_y| > Mm$ for $x^2 + y^2 \geqslant R^2$. When C_i are large enough, arcs K_i will lie outside the cirtle $x^2 + y^2 = R^2$ and if $x(t), y(t)$ is any solution of (1), the initial point of which belongs to K, then the following inequality holds

$$\frac{dG\big(x(t), y(t); q_i, r_i\big)}{dt} = H_x\big(x(t), y(t)\big)\big(r_i + p_1(t)\big) + H_y\big(x(t), y(t)\big)\big(-q_i + p_2(t)\big) + \\ + q_i p_1(t) + r_i p_2(t) > (|H_x| + |H_y|)\,m - M > 0 \quad (i = 1, \ldots, 5)\,.$$

It is easy to notice now that the solution $x(t), y(t)$ cannot enter the domain D crossing K, so the curve $L_1(t)$ lies outside the domain D.

D will include the circle $x^2 + y^2 = \varrho^2$ if the constants C_i are sufficiently large. Hence we can find a number r_0 such that (A) holds.

Proof of (B). Let us introduce polar functions $r(t), \varphi(t)$:

$$r(t) = \{x^2(t, u, v) + y^2(t, u, v)\}^{1/2}\,, \qquad \varphi(t) = \operatorname{arctg} \frac{y(t, u, v)}{x(t, u, v)}\,.$$

$\varphi(t)$ is a solution of equation

$$\text{(8)} \qquad \dot{\varphi} = \frac{H_x(r, \varphi)}{r}\cos\varphi + \frac{H_y(r, \varphi)}{r}\sin\varphi + \frac{p_2(t)}{r}\cos\varphi - \frac{p_1(t)}{r}\sin\varphi$$

where

$$H_x(r, \varphi) = H_x(r\cos\varphi, r\sin\varphi)\,, \qquad H_y(r, \varphi) = H_y(r\cos\varphi, r\sin\varphi)\,.$$

Let ε, η be arbitrarily small constants. From (A) we conclude that if $u^2+v^2 \geqslant r_0^2$ then $r(t) > \varrho$ for $t \in \langle t_0, t_1\rangle$. A constant ϱ is such that

$$\text{(9)} \qquad \frac{|H_x(r,\varphi)|}{r} < A+\eta\,, \qquad \frac{|H_y(r,\varphi)|}{r} < B+\eta$$

and

$$\text{(10)} \qquad \frac{|p_2(t)\cos\varphi - p_1(t)\sin\varphi|}{r} < \varepsilon$$

for $r > \varrho$ and any φ.

(9) and (10) imply the inequality

$$\text{(11)} \qquad \dot\varphi < A_1|\cos\varphi| + B_1|\sin\varphi|\,, \quad A_1 = A+\varepsilon+\eta\,, \quad B_1 = B+\varepsilon+\eta\,.$$

(3), (4) and (8) imply that $\dot\varphi$ is positive if ϱ is sufficiently large. So $\varphi(t)$ is inversible and let $\tau(\varphi)$ be its inverse function. For $t \in \langle t_0, t_1\rangle$ we have the inequality

$$\frac{d\tau}{d\varphi} > \frac{1}{A_1|\cos\varphi| + B_1|\sin\varphi|}\,.$$

Integrating it in the interval $[0, 2\pi]$ we obtain

$$t_1 - t_0 = \tau(2\pi) - \tau(0) > \frac{4}{\sqrt{A_1^2+B_1^2}} \ln \frac{A_1+B_1+\sqrt{A_1^2+B_1^2}}{A_1+B_1-\sqrt{A_1^2+B_1^2}}\,.$$

We can choose r_0 so large, that the numbers A_1, B_1 will be arbitrary close to A, B, so the following inequality will be satisfied

$$\frac{1}{\sqrt{A_1^2+B_1^2}} \ln \frac{A_1+B_1+\sqrt{A_1^2+B_1^2}}{A_1+B_1-\sqrt{A_1^2+B_1^2}} > \frac{1}{\sqrt{A^2+B^2}} \ln \frac{A+B+\sqrt{A^2+B^2}}{A+B-\sqrt{A^2+B^2}} - \delta\,.$$

This inequality completes the proof of (B).

Proof of Theorem 1. Assume additionally the uniqueness of solutions of (1). Let $x(t,u,v)$, $y(t,u,v)$ be a solution of (1) satisfying the initial conditions $x(0,u,v) = u$, $y(0,u,v) = v$. Let us introduce vector fields C_0, C_t, $(0 < t \leqslant \omega)$ by ascribing the vectors

$$V_0(u,v) = \big(-H_y(u,v) + p_1(0),\ H_x(u,v) + p_2(0)\big)$$

$$V_t(u,v) = \frac{1}{t}\big(x(t,u,v) - u,\ y(t,u,v) - v\big)$$

to every point (u,v) of plane, respectively.

To demonstrate the theorem it is sufficient to show that C_ω has a singular point say (u_0, v_0), for which $V_\omega(u_0, v_0) = 0$, that is

$$x(\omega, u_0, v_0) = u_0 = x(0, u_0, v_0); \quad y(\omega, u_0, v_0) = v_0 = y(0, u_0, v_0)\,.$$

For this purpose it will be proved that the index of Γ relative to C_ω, $\mathrm{Ind}(\Gamma, C_\omega)$, is equal to $+1$, where Γ denotes a certain Jordan curve of plane ([1], p. 337). If the field $C_t (0 < t \leqslant \omega)$ has no singular points on Γ and $\mathrm{Ind}(\Gamma, C_0) = 1$, then $\mathrm{Ind}(\Gamma, C_\omega) = 1$ because $C_0 = \lim_{t\to 0+} C_t$ and one can pass continuously from the field C_0 to C_ω.

The curve Γ is chosen among the curves of the family K_c: $H(x, y) = C$ (C-constant). As it has been shown, K_c are Jordan curves. By (4) and boundednes of $p_1(t), p_2(t)$ vectors $V_0(u, v)$, $(u, v) \in K_c$ are "almost tangent" to K_c if C is large enough, hence $\mathrm{Ind}(K_c, C_0) = 1$. To finish the proof it will be sufficient to demonstrate that for large C, the field C_t has no singular points on K_c.

Suppose, if possible, that for every constant C there exists a point $(u_0, v_0) \in K_c$ and a number $t_0 \leqslant \omega < \omega_0 - \delta$ such that $V_{t_0}(u_0, v_0) = 0$; that is

$$x(t_0, u_0, v_0) = x(0, u_0, v_0), \quad y(t_0, u_0, v_0) = y(0, u_0, v_0),$$

where $x = x(t, u_0, v_0)$, $y = y(t, u_0, v_0)$ are the parametric equations of the integral curve $L(t)$ which starts from the point (u_0, v_0). Obviously $L(t)$ is a closed curve. Let $L_1(t)$ be defined as previously in the lemma. Since the arc $L(t)$ is closed $t_1 \leqslant t_0$. By (B) a constant C can be chosen so large that $t_1 > \omega_0 - \delta > t_0$. But it contradicts $t_1 \leqslant t_0$. Thus Theorem 1 under the assumption of the uniqueness is proved. This assumption can be omitted. To prove that let us consider the sequence of functions $\{H^n(x, y)\}$ of class C^2, such that $H^n_x(x, y), H^n_y(x, y)$ converge uniformly to $H_x(x, y), H_y(x, y)$. Let $H^n(x, y)$ satisfies the assumptions of Theorem 1, then the system

$$\text{(1}n\text{)} \qquad \dot{x} = -H^n_y(x, y) + p_1(t); \quad \dot{y} = H^n_x(x, y) + p_2(t)$$

possesses the unique solutions and there exists the periodic solution $\{x_n(t), y_n(t)\}$ say with the period ω. The functions $\{x_n(t), y_n(t)\}$ are equibounded and equicontinuous because the derivatives $\dot{x}_n(t), \dot{y}_n(t)$ are bounded which follows from the inequalities

$$|\dot{x}_n(t)| \leqslant \left|H^n_y(x_n(t), y_n(t))\right| + p; \qquad |\dot{y}_n(t)| \leqslant \left|H^n_x(x_n(t), y_n(t))\right| + p.$$

There exists a subsequence of sequence $\{x_n(t), y_n(t)\}$ which is uniformly convergent in the interval $\langle 0, \omega\rangle$. The limit functions $x(t), y(t)$ are the solution of (1) and they are periodic with the period ω.

3. Theorem 2. *Suppose that the assumptions of Theorem* 1 *are satisfied except* (5) *which is replaced by the assumption*

$$\text{(12)} \qquad \frac{H_x(x, y)x + H_y(x, y)y}{x^2 + y^2} < h\left(\frac{x}{r}, \frac{y}{r}\right) \quad \text{for} \quad x^2 + y^2 \geqslant r_0^2,$$

where $h(z, w)$ is continuous and positive for $z^2 + w^2 = 1$.

If the period ω of functions p_1, p_2 satisfies the inequality

$$\omega < \omega_0 = \int_0^{2\pi} \frac{d\varphi}{h(\cos\varphi, \sin\varphi)}$$

then the system (1) *has the periodic solution with the period ω.*

Proof. We will prove, that if the assumption (5) in Lemma is replaced by (12), then (B) will be satisfied. The remaining part of the proof of Theorem 2 is the same as in the proof of Theorem 1.

Let us assume (12). Then for $u^2+v^2 \geqslant r_0^2$ (r_0 is large enough) the function $\varphi(t)$ satisfies the inequality

$$\dot{\varphi} < h(\cos\varphi, \sin\varphi)+\varepsilon$$

which is deduced like (11). So we have $t_1-t_0 = \tau(2\pi)-\tau(0) > T(\varepsilon)$, where

$$T(\varepsilon) = \int_0^{2\pi} \frac{d\varphi}{h(\cos\varphi, \sin\varphi)+\varepsilon}\,.$$

The number ε can be made small if r_0 is large enough. Since the function $T(\varepsilon)$ is continuous and $\lim_{\varepsilon\to 0} T(\varepsilon) = \omega_0$ we have (B).

Corollary (compare [2], Theorem 2). *If the function $g(x)$ is continuous and satisfies the conditions*

$$xg(x) > 0 \quad \text{for} \quad x \neq 0\,, \qquad \lim_{|x|\to\infty} |g(x)| = \infty\,,$$

$$\limsup_{|x|\to\infty} \frac{g(x)}{x} \leqslant \frac{4\pi^2}{T_0^2}\,,$$

the function $p(t)$ is periodic with the period $\omega < T_0$ and continuous, then the equation

(13) $$\ddot{x}+g(x) = p(t)$$

has at least one periodic solution with the period ω.

Proof. We replace (13) by the system

$$\dot{x} = -y\,, \qquad \dot{y} = g(x)-p(t)$$

and we put $p_1(t)\equiv 0$, $p_2(t) = -p(t)$, $H(x,y) = \frac{1}{2}y^2+G(x)$ $\left(G(x) = \int_0^x g(s)\,ds\right)$ and $h(z,w) = \frac{4\pi^2}{T_0^2}z^2+w^2$. It is sufficient now to examine, that all assumptions of Theorem 2 are satisfied.

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