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On a certain sequence of ordinary differential equations

1. Let Λ denote the family of real-valued functions fulfilling the Lipschitz condition in the interval $\langle 0,1\rangle$ with the constant 1, i.e. $\Lambda = \{\lambda(x): |\lambda(\bar{x}) - \lambda(\bar{\bar{x}})| \leqslant |\bar{x} - \bar{\bar{x}}|,\ \bar{x}, \bar{\bar{x}} \in \langle 0,1\rangle\}$.

Moreover, $P = \{(x,y): x \in \langle 0,1\rangle,\ -\infty < y < +\infty\}$.

Theorem. *There exists a sequence of differential equations*

$$(\mathrm{I}_k) \qquad y' = F_k(x,y)\,, \quad k = 1,2,\ldots$$

such that

1° $F_k(x,y)$ *are continuous and* $|F_k(x,y)| \leqslant 1$ *in* P,

2° $\lim\limits_{k\to\infty} F_k(x,y) = 0$ *almost everywhere, i.e. except a set of Lebesgue's plane measure zero,*

3° *for every function* $\lambda(x) \in \Lambda$ *there exists a sequence of indices* $\{a_k\}$ *and a sequence* $\{\varphi_{a_k}(x)\}$ *of solutions of the equations* (I_{a_k}) *such that* $\{\varphi_{a_k}(x)\}$ *converges uniformly to* $\lambda(x)$ *in* $\langle 0,1\rangle$.

2. We shall first prove the following

Lemma. *For every function* $\lambda(x) \in \Lambda$ *there exists a sequence of differential equations*

$$(1_\nu) \qquad y' = f_\nu(x,y) \quad \nu = 1,2,\ldots$$

such that

1° *functions* $f_\nu(x,y)$ *are continuous and* $|f_\nu(x,y)| \leqslant 1$ *in* P,

2° $m(\operatorname{supp} f_\nu(x,y)) < 1/\nu 2^\nu$ *.

3° *for every* ν *there exists a solution* $\varphi_\nu(x)$ *of the equation* (1_ν) *such that* $\{\varphi_\nu(x)\}$ *converges uniformly in the interval* $\langle 0,1\rangle$ *to the function* $\lambda(x)$.

Proof. Since $\left(\lambda(x) - 3/\nu 2^{\nu+1}\right) \in \Lambda$, there exists [2] a sequence of functions $\{\alpha_n^\nu(x)\}$ of class C^1 uniformly convergent in the interval $\langle 0,1\rangle$ to $\lambda(x) - 3/\nu 2^{\nu+1}$ and satisfying the inequality $\left|(\alpha_n^\nu(x))'\right| \leqslant 1$ for $x \in \langle 0,1\rangle$ and $n, \nu = 1,2,\ldots$

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* The $\operatorname{supp} \alpha(x)$ denotes a support of a (continuous) function $\alpha(x)$, i.e. the closure of the set of points for which $\alpha(x) \neq 0$. $m(A)$ denotes the Lebesgue's plane measure of a set A.

Let $a^{\nu}_{N(\nu)}(x)$ be a fixed function of the sequence $\{a^{\nu}_{n}(x)\}$ satisfying the inequality

$$\left|a^{\nu}_{N(\nu)}(x)-\left(\lambda(x)-3/\nu 2^{\nu+1}\right)\right|<1/\nu 2^{\nu+1} \quad \text{for} \quad x\in\langle 0,1\rangle .$$

We define the function $\tilde{f}_{\nu}(x, y)$ as follows

$$\tilde{f}_{\nu}(x,y)=\begin{cases} 0 & \text{for} \quad (x,y)\in P \text{ and } y\geqslant\lambda(x)-1/\nu 2^{\nu} \\ \left(a^{\nu}_{N(\nu)}(x)\right)' & \text{for} \quad (x,y)\in P \text{ and } y=a^{\nu}_{N(\nu)}(x) \\ 0 & \text{for} \quad (x,y)\in P \text{ and } y\leqslant\lambda(x)-1/\nu 2^{\nu-1} . \end{cases}$$

Let $f_{\nu}(x, y)$ be a continuous extension of the function $\tilde{f}_{\nu}(x, y)$ on whole set P satisfying inequality $|f_{\nu}(x, y)|\leqslant 1$. It is easy to see that $f_{\nu}(x, y)$ satisfies 1°. $f_{\nu}(x, y)$ satisfies also the condition 2°, because

$$\operatorname{supp} f_{\nu}(x,y)\subset\{(x,y)\in P\colon\ \lambda(x)-1/\nu 2^{\nu-1}\leqslant y\leqslant\lambda(x)-1/\nu 2^{\nu}\} .$$

Setting $\varphi_{\nu}(x)=a^{\nu}_{N(\nu)}(x)$, we obtain a sequence required in 3°.

3. Proof of the theorem. The family Λ is a subset of the space of continuous real-valued functions in the interval $\langle 0, 1\rangle$. That space is separable if to define the norm $\|f\|=\max\limits_{\langle 0,1\rangle}|f(x)|$. Therefore there exists a sequence of functions $\lambda^{n}(x)$ dense in Λ, $\lambda^{n}(x)\in\Lambda$.

For every function $\lambda^{n}(x)$ we form a sequence of differential equations

$$y'=f^{n}_{\nu}(x,y)$$

such as in the lemma; in particular, the property 2° has now a form

$$m\left(\operatorname{supp} f^{n}_{\nu}(x,y)\right)\leqslant 1/\nu 2^{\nu}, \qquad n=1,2,\ldots$$

We range all functions $f^{n}_{\nu}(x, y)$ for $\nu\geqslant n$ in a single sequence setting $F_1(x, y) = f^{1}_{1}(x, y)$, $F_2(x, y) = f^{2}_{1}(f(x, y)$, $F_3(x, y) = f^{2}_{2}(x, y)$, $F_4(x, y) = f^{3}_{1}(x, y)$ etc.

We shall show that the sequence $y'=F_k(x, y)$ is the required sequence. The condition 1° is obviously satisfied. Using the following theorem ([1], p. 294) we shall prove the condition 2°:

If $\{g_n(x)\}$ is a sequence of measurable functions, non-negative in a set A and

$$\sum_{n=1}^{\infty}\int_{A} g_n(x)\,dx<+\infty ,$$

then $\lim\limits_{n\to\infty} g_n(x)=0$ almost everywhere in A.

Indeed,

$$\sum_{k=1}^{\infty}\int_{P}|F_k(x,y)|\,dxdy=\sum_{k=1}^{\infty}\int_{\operatorname{supp}F_k(x,y)}|F_k(x,y)|\,dxdy\leqslant$$

$$\leqslant\sum_{k=1}^{\infty} m\left(\operatorname{supp}F_k(x,y)\right)=\sum_{\substack{n=1\\ \nu\geqslant n}}^{\infty} m\left(\operatorname{supp} f^{n}_{\nu}(x,y)\right)\leqslant\sum_{\nu=1}^{\infty}\nu\left(\frac{1}{\nu 2^{\nu}}\right)<+\infty .$$

Hence $\lim_{k\to\infty} F_k(x, y) = 0$ almost everywhere in P.

Let $\lambda(x) \in \Lambda$ be a fixed function. The density of the sequence $\{\lambda^n(x)\}$ implies that for every fixed m there exists a index $\beta(m) = \beta_{N(m)}$ such that

$$(1) \qquad |\lambda^{\beta(m)}(x) - \lambda(x)| < 1/2m \qquad (0 \leqslant x \leqslant 1)\,.$$

Moreover, the point 3° of the lemma implies that there exists a solution $\varphi_{\nu(m)}^{\beta(m)}(x)$ $\big(\nu(m) \geqslant \beta(m)\big)$ of the equation $y' = f_{\nu(m)}^{\beta(m)}(x, y)$ satisfying the inequality

$$(2) \qquad |\varphi_{\nu(m)}^{\beta(m)}(x) - \lambda^{\beta(m)}(x)| < 1/2m \qquad (0 \leqslant x \leqslant 1)\,.$$

(1) and (2) imply

$$|\varphi_{\nu(m)}^{\beta(m)}(x) - \lambda(x)| < 1/2m \qquad (0 \leqslant x \leqslant 1)\,.$$

Since $\nu(m) \geqslant \beta(m)$, the sequence $\{f_{\nu(m)}^{\beta(m)}(x, y)\}$ $m = 1, 2, \ldots$ is a subsequence of the sequence $\{F_k(x, y)\}$. We denote by $\{F_{a_k}(x, y)\}$ the sequence so chosen. It is easy to see that the corresponding sequence $\{\varphi_{a_k}(x)\} = \{\varphi_{\nu(m)}^{\beta(m)}(x)\}$ converges uniformly to $\lambda(x)$ in $\langle 0, 1\rangle$.

REFERENCES

[1] R. **Sikorski**, *Funkcje rzeczywiste I*, Warszawa 1958.

[2] J. **Szarski**, *Sur une méthode d'approximation des fonctions*, Ann. Soc. Polon. Math. 20 (1947), p. 121-125.