

**THE NATURAL OPERATORS LIFTING  $k$ -PROJECTABLE  
VECTOR FIELDS TO PRODUCT-PRESERVING BUNDLE  
FUNCTORS ON  $k$ -FIBERED MANIFOLDS**

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**Abstract.** For any product-preserving bundle functor  $F$  defined on the category  $k\text{-}\mathcal{FM}$  of  $k$ -fibered manifolds, we determine all natural operators transforming  $k$ -projectable vector fields on  $Y \in \text{Ob}(k\text{-}\mathcal{FM})$  to vector fields on  $FY$ . We also determine all natural affinors on  $FY$ . We prove a composition property analogous to that concerning Weil bundles.

**0. Preliminaries.** The classical results by Kainz and Michor [6], Luciano [11] and Eck [3] read that the product-preserving bundle functors on the category  $\mathcal{Mf}$  of manifolds are just Weil bundles, [17]. Let us remind Kolář's result [7].

For a bundle functor  $F$  on  $\mathcal{Mf}$ , denote by  $\mathcal{F}$  the flow operator lifting vector fields to  $F$ . Further, consider an element  $c$  of a Weil algebra  $A$  and let  $L(c)_M : TT^AM \rightarrow TT^AM$  denote the natural affinor by Koszul ([7], [8]). Then we have a natural operator  $L(c)_M \circ \mathcal{T}^A : TM \rightsquigarrow TT^AM$  lifting vector fields on a manifold  $M$  to a Weil bundle  $T^AM$ .

The Lie algebra associated to the Lie group  $\text{Aut}(A)$  of all algebra automorphisms of  $A$  is identified with the algebra of derivations  $\text{Der}(A)$  of  $A$ . For any  $D \in \text{Der}(A)$  consider its one-parameter subgroup  $\delta(t) \in \text{Aut}(A)$ . It determines the vector field  $D_M = \frac{d}{dt}_0 \delta(t)_M$  on  $T^AM$ , where we identify Weil algebra homomorphisms with the corresponding natural transformations. Finally, we

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obtain a natural operator  $\Lambda_{D,M} : TM \rightsquigarrow TT^A M$  defined by  $\Lambda_{D,M}(X) = D_M$  for any vector field  $X$  on  $M$ . Then Kolář's result reads as follows.

*All natural operators  $TM \rightsquigarrow TT^A M$  are of the form  $L(c)_M \circ \mathcal{T}^A + \Lambda_{D,M}$  for some  $c \in A$  and  $D \in \text{Der}(A)$ .*

Let us remind some results concerning product-preserving bundle functors on the category  $\mathcal{FM}$  of fibered manifolds, [12], [2], [16]. They are just of the form  $T^\mu$  for a homomorphism  $\mu : A \rightarrow B$  of Weil algebras. Bundle functors  $T^\mu$  are defined as follows. Let  $i, j : \mathcal{M}f \rightarrow \mathcal{FM}$  be functors defined by  $i(M) = id_M : M \rightarrow M$  and  $j(M) = (M \rightarrow pt)$  for a manifold  $M$  and the single-point manifold  $pt$ . If  $F : \mathcal{FM} \rightarrow \mathcal{FM}$  preserves the product then so do  $G^F = F \circ i$  and  $H^F = F \circ j$  and so there are Weil algebras  $A$  and  $B$  such that  $G^F = T^A$  and  $H^F = T^B$ . Further, there is an obvious natural identity transformation  $\tau_M : i(M) \rightarrow j(M)$  and thus we have a natural transformation  $\mu_M = F\tau_M : T^A M \rightarrow T^B M$  corresponding to a Weil algebra homomorphism  $\mu : A \rightarrow B$ . Then the functor  $T^\mu$  can be defined as the pull-back  $T^A M \times_{T^B M} T^B Y$  with respect to  $\mu$  and  $T^B p$  for a fibered manifold  $p : Y \rightarrow M$ . Then  $F = T^\mu$  modulo a natural equivalence.

Let  $\bar{F}$  be another product-preserving bundle functor on  $\mathcal{FM}$ . Then the result of [12] also yields natural transformations  $\eta : F \rightarrow \bar{F}$  in the form of couples of  $(\mu, \bar{\mu})$ -related natural transformations  $\nu = \eta \circ i : T^A \rightarrow T^{\bar{A}}$  and  $\rho : \eta \circ j : T^B \rightarrow T^{\bar{B}}$  for a Weil algebra homomorphisms  $\nu : A \rightarrow \bar{A}$  and  $\sigma : B \rightarrow \bar{B}$ .

For a bundle functor  $F$  on  $\mathcal{FM}$ , denote by  $\mathcal{F}$  the flow operator lifting projectable vector fields to  $F$ . Further, consider an element  $c$  of  $A$  and let  $L(c)_Y : TT^\mu Y \rightarrow TT^\mu Y$ ,  $L(c)_Y(y_1, y_2) = (L(c)_M(y_1), L(\mu(c))_Y(y_2))$ ,  $(y_1, y_2) \in TT^\mu Y = TT^A M \times_{TT^B M} TT^B Y$  be the modification of the Koszul affiner. Then we have a natural operator  $L(c)_Y \circ \mathcal{T}^\mu : T_{proj} Y \rightsquigarrow TT^\mu Y$  lifting projectable vector fields on a fibered manifold  $Y$  to  $T^\mu Y$  for a Weil algebra homomorphism  $\mu : A \rightarrow B$ .

The Lie algebra associated to the Lie group  $Aut(\mu) = \{(\nu, \rho) \in Aut(A) \times Aut(B) \mid \rho \circ \mu = \mu \circ \nu\}$  of all automorphisms of  $\mu$  is identified with the algebra of derivations  $Der(\mu) = \{D = (D_1, D_2) \in Der(A) \times Der(B) \mid D_2 \circ \mu = \mu \circ D_1\}$  of  $\mu$ . For any  $D \in Der(\mu)$  consider its one-parameter subgroup  $\delta(t) \in Aut(\mu)$ . It determines the vector field  $D_Y = \frac{d}{dt}_0 \delta(t)_Y$  on  $T^\mu Y$ , where we identify homomorphisms of  $\mu$  with the corresponding natural transformations. Finally, we obtain a natural operator  $\Lambda_{D,Y} : T_{proj} Y \rightsquigarrow TT^\mu Y$  defined by  $\Lambda_{D,Y}(X) = D_Y$  for any projectable vector field  $X$  on  $Y$ . Then a result of Tomáš [16] reads

All natural operators  $T_{proj}Y \rightsquigarrow TT^\mu Y$  are of the form  $L(c)_Y \circ T^\mu + \Lambda_{D,Y}$  for some  $c \in A$  and  $D \in Der(\mu)$ .

Let us recall the concept of  $k$ -fibered manifolds. It is a sequence of surjective submersions

$$(1) \quad Y = Y_k \xrightarrow{p_k} Y_{k-1} \xrightarrow{p_{k-1}} \dots \xrightarrow{p_1} Y_0$$

between manifolds. Given another  $k$ -fibered manifold  $\bar{Y} = \bar{Y}_k \xrightarrow{\bar{p}_k} \bar{Y}_{k-1} \xrightarrow{\bar{p}_{k-1}} \dots \xrightarrow{\bar{p}_1} \bar{Y}_0$ , a map  $f : Y \rightarrow \bar{Y}$  is called a morphism of  $k$ -fibered manifolds if there are the so-called underline maps  $f_j : X_j \rightarrow \bar{X}_j$  for  $j = 0, \dots, k - 1$  such that  $f_{j-1} \circ p_j = \bar{p}_j \circ f_j$  for  $j = 1, \dots, k$ , where  $f_k = f$ . Thus we have the category  $k - \mathcal{FM}$  of  $k$ -fibered manifolds which is local and admissible in the sense of [8]. Clearly, the category  $1 - \mathcal{FM}$  of 1-fibered manifolds coincides with the category  $\mathcal{FM}$  of fibered manifolds.

Let us remind some results concerning product-preserving bundle functors on the category  $k - \mathcal{FM}$  of  $k$ -fibered manifolds, [13]. They are just of the form  $T^\mu$  for a sequence

$$(2) \quad \mu = (A_k \xrightarrow{\mu^k} A_{k-1} \xrightarrow{\mu^{k-1}} \dots \xrightarrow{\mu^1} A_0)$$

of  $k$  Weil algebra homomorphisms. Bundle functors  $T^\mu$  are defined as follows. Let  $i^{[l]} : \mathcal{M}f \rightarrow k - \mathcal{FM}$  for  $l = 0, \dots, k$  be a sequence of functors defined by  $i^{[l]}(M) = pt_M^{[l+1]} = (M \xrightarrow{id_M} M \xrightarrow{id_M} \dots \xrightarrow{id_M} M \rightarrow pt \rightarrow \dots \rightarrow pt)$ ,  $k - l$  times of the single-point manifold  $pt$ , and  $i^{[l]}(f) = f$ . If  $F : k - \mathcal{FM} \rightarrow \mathcal{FM}$  preserves the product then so do  $G^{l,F} = F \circ i^{[l]}$  and so there are Weil algebras  $A_l$  such that  $G^{l,F} = T^{A_l}$  for  $l = 0, \dots, k$ . Further, there are obvious identity natural transformations  $\tau_M^l : i^{[l]}(M) \rightarrow i^{[l-1]}(M)$  and thus we have a sequence of natural transformations  $\mu_M^l = F\tau_M^l$  corresponding to a sequence  $\mu = (A_k \xrightarrow{\mu^k} A_{k-1} \xrightarrow{\mu^{k-1}} \dots \xrightarrow{\mu^1} A_0)$  of Weil algebra homomorphisms. For any  $k$ -fibered manifold  $Y$  of the form (1) we have

$$(3) \quad \begin{aligned} T^\mu Y = \{ & y = (y_k, y_{k-1}, \dots, y_0) \in T^{A_k} Y_0 \times T^{A_{k-1}} Y_1 \times \dots \times T^{A_0} Y_k \mid \\ & \mu_{Y_l}^{k-l}(y_{k-l}) = T^{A_{k-l-1}} p_{l+1}(y_{k-l-1}), \quad l = 0, \dots, k - 1 \}. \end{aligned}$$

For a  $k - \mathcal{FM}$ -map  $f : Y \rightarrow \bar{Y}$ ,  $T^\mu f : T^\mu Y \rightarrow T^\mu \bar{Y}$  is the restriction and correstriction of  $T^{A_k} f_0 \times T^{A_{k-1}} f_1 \times \dots \times T^{A_0} f_k$ . Then  $F = T^\mu$  modulo a natural equivalence.

Let  $\bar{F}$  be another product-preserving bundle functor on  $k - \mathcal{FM}$ . Then the results of [13] also yield natural transformations  $\eta : F \rightarrow \bar{F}$  in the form

of sequences  $\nu = (\nu^k, \dots, \nu^0)$  of  $(\mu, \bar{\mu})$ -related natural transformations  $\nu^l = \eta \circ i^{[l]} : T^{A_l} \rightarrow T^{\bar{A}_l}$  for Weil algebra homomorphisms  $\nu^l : A_l \rightarrow \bar{A}_l$ .

We shall investigate  $k$ -projectable vector fields. A vector field  $X$  on a  $k$ -fibered manifold  $Y$  of the form (1) is called  $k$ -projectable if there are vector fields  $X_l$  on  $Y_l$  for  $l = 0, \dots, k - 1$  which are related to  $X$  by the respective compositions of projections of  $Y$ . The flow of  $X$  is formed by local  $k - \mathcal{FM}$ -isomorphisms. The space of all  $k$ -projectable vector fields on  $Y$  will be denoted by  $\mathcal{X}_{k-proj}(Y)$ .

Natural operators lifting vector fields are used in practically each paper in which the problem of prolongations of geometric structures was studied. For example A. Morimoto [15] used liftings of functions and vector fields has been to define the complete lifting of connections. That is why such natural operators are classified in [4], [7], [16] and other papers (over 100 references). For example, in the case of the tangent bundle  $TM$  of a manifold  $M$  (in our notation,  $k = 0$ ), any natural operator lifting vector fields from  $M$  to  $TM$  is a linear combination of the complete lifting, the vertical lifting and the Liouville (dilatation) vector field.

A torsion of a connection  $\Gamma$  on  $TM$  is the Nijenhuis bracket  $[\Gamma, J]$  of  $\Gamma$  with the almost tangent structure  $J$  on  $TM$ . This fact has been generalized in [9] in such a way that a torsion of a connection  $\Gamma$  with respect to a natural affnor  $A$  is  $[\Gamma, A]$ . Thus natural affnors can be used to study torsions of connections. That is why they have been classified in [1], [5], [10] and other papers (over 20 references). For example, any natural affnor on  $TM$  is a linear combination of the identity affnor and the almost tangent structure on  $TM$ .

**1. Some properties of product preserving bundle functors on  $k - \mathcal{FM}$ .** According to the Weil theory [6], for Weil algebras  $A$  and  $B$  there is the canonical identification  $T^A \circ T^B M = T^{B \otimes A} M$ . We generalize this fact on  $k - \mathcal{FM}$ . This extends the respective result of Tomáš's [16].

Consider  $T^\mu Y$  in the form (3), where  $\mu$  is of the form (2) and  $Y$  is of the form (1). It is easy to see that  $T^\mu Y$  is a  $k$ -fibered manifold if we consider it in the form

$$(4) \quad T^\mu Y = T^{\mu^{[k]}} Y_{[k]} \rightarrow T^{\mu^{[k-1]}} Y_{[k-1]} \rightarrow \dots \rightarrow T^{\mu^{[0]}} Y_{[0]} ,$$

where  $\mu^{[l]} = (A_k \xrightarrow{\mu^k} A_{k-1} \xrightarrow{\mu^{k-1}} \dots \xrightarrow{\mu^{k-l+1}} A_{k-l})$  is the truncation of  $\mu$  (it is a sequence of  $l$  Weil algebra homomorphisms) and  $Y_{[l]} = Y_l \xrightarrow{p_l} Y_{l-1} \xrightarrow{p_{l-1}} \dots \xrightarrow{p_1} Y_0$  is the truncation of  $Y$  (it is an  $l - \mathcal{FM}$ -object) and where  $T^{\mu^{[l]}} Y_{[l]}$  is defined as in (3) (in particular,  $T^{\mu^{[0]}} Y_{[0]} = T^{A_0} Y_0$ ). Here the arrows in (4) are the restrictions and correstrictions of the obvious projections  $T^{A_k} Y_0 \times \dots \times$

$T^{A_{k-l}}Y_l \rightarrow T^{A_k}Y_0 \times \dots \times T^{A_{k-l+1}}Y_{l-1}$ . Then  $T^\mu : k - \mathcal{FM} \rightarrow \mathcal{FM}$  is a functor  $k - \mathcal{FM} \rightarrow k - \mathcal{FM}$ . Thus we can compose product-preserving bundle functors on  $k - \mathcal{FM}$ .

PROPOSITION 1. Let  $T^\mu, T^{\bar{\mu}} : k - \mathcal{FM} \rightarrow \mathcal{FM}$  be product-preserving bundle functors corresponding to sequences  $\mu$  and  $\bar{\mu}$  of the form (2). Then  $T^\mu \circ T^{\bar{\mu}} = T^{\bar{\mu} \otimes \mu}$ , where (of course)  $\bar{\mu} \otimes \mu = (\bar{A}_k \otimes A_k \xrightarrow{\bar{\mu}^k \otimes \mu^k} \bar{A}_{k-1} \otimes A_{k-1} \xrightarrow{\bar{\mu}^{k-1} \otimes \mu^{k-1}} \dots \xrightarrow{\bar{\mu}^1 \otimes \mu^1} \bar{A}_0 \otimes A_0)$ .

PROOF. Let  $\tilde{\mu} = (\tilde{A}_k \xrightarrow{\tilde{\mu}^k} \tilde{A}_{k-1} \xrightarrow{\tilde{\mu}^{k-1}} \dots \xrightarrow{\tilde{\mu}^1} \tilde{A}_0)$  be the sequence of the form (2) corresponding to the composition  $T^\mu \circ T^{\bar{\mu}}$ . It can be computed as described in Section 0. Thus by the mentioned Weil theory [6], there is  $\tilde{A}_l = \bar{A}_l \otimes A_l$  (as there is the identification  $\tilde{A}_l = T^{A_l} \circ T^{\bar{A}_l}(\mathbf{R}) = T^{\bar{A}_l \otimes A_l}(\mathbf{R}) = \bar{A}_l \otimes A_l$ ). This identification is  $(\tilde{\mu}, \bar{\mu} \otimes \mu)$ -related.  $\square$

We describe some special case of  $T^\mu$ . Let  $\mu$  be of the form (2), where  $A_k = A_{k-1} = \dots = A_0 = A$  and  $\mu^l = id_A$  for  $l = 1, \dots, k$ . We will write  $id^A$  for such  $\mu$ . Then  $T^{id^A}Y = T^AY$ . In particular,  $T^{id}Y = TY$ , where  $id = id^{\mathbf{D}}$  and  $\mathbf{D}$  is the Weil algebra of dual numbers.

**2. Natural vector fields on bundle functors  $T^\mu$ .** Consider a sequence  $\mu$  of the form (2). The group

$$Aut(\mu) = \{ \nu = (\nu^k, \nu^{k-1}, \dots, \nu^0) \in Aut(A_k) \times Aut(A_{k-1}) \times \dots \times Aut(A_0) \mid \nu^{l-1} \circ \mu^l = \mu^l \circ \nu^l, l = 1, \dots, k \}$$

of all automorphisms of  $\mu$  is a closed subgroup in  $Aut(A_k) \times Aut(A_{k-1}) \times \dots \times Aut(A_0)$ . Thus  $Aut(\mu)$  is a Lie group. Let

$$Der(\mu) = \{ D = (D^k, D^{k-1}, \dots, D^0) \in Der(A_k) \times Der(A_{k-1}) \times \dots \times Der(A_0) \mid D^{l-1} \circ \mu^l = \mu^l \circ D^l, l = 1, \dots, k \}$$

be the Lie algebra of all derivations of  $\mu$ .

PROPOSITION 2. Let  $Lie(Aut(\mu))$  be the Lie algebra of the Lie group  $Aut(\mu)$  of all automorphisms of  $\mu$  of the form (2). Then  $Lie(Aut(\mu)) = Der(\mu)$ .

PROOF. We know that  $Lie(Aut(A)) = Der(A)$  for any Weil algebra  $A$  ([7]). Consequently, the proposition follows directly from the application of exponential mapping concept.  $\square$

Let us recall that a natural operator  $\Lambda_Y : T_{k-proj}Y \rightsquigarrow TT^\mu Y$  is a system of regular  $k - \mathcal{FM}$ -invariant operators

$$\Lambda_Y : \mathcal{X}_{k-proj}(Y) \rightarrow \mathcal{X}(T^\mu Y)$$

for any  $k - \mathcal{FM}$ -object  $Y$ . The  $k - \mathcal{FM}$ -invariance means that for any  $k - \mathcal{FM}$ -objects  $Y, \bar{Y}$ , any  $k$ -projectable vector fields  $X \in \mathcal{X}_{k-proj}(Y)$  and  $\bar{X} \in \mathcal{X}_{k-proj}(\bar{Y})$  and any  $k - \mathcal{FM}$ -map  $f : Y \rightarrow \bar{Y}$ , if  $X$  and  $\bar{X}$  are  $f$ -related (i.e.  $Tf \circ X = \bar{X} \circ f$ ) then  $\Lambda_Y(X)$  and  $\Lambda_{\bar{Y}}(\bar{X})$  are  $T^\mu f$ -related. The regularity means that  $\Lambda_Y$  transforms smoothly parametrized families of  $k$ -projectable vector fields into smoothly parametrized families of vector fields.

A natural operator  $\Lambda_Y : T_{k-proj}Y \rightsquigarrow TT^\mu Y$  is called absolute (or a natural vector field on  $T^\mu$ ) if  $\Lambda_Y$  is a constant function for any  $Y \in Obj(k - \mathcal{FM})$ .

Proposition 2 enables us to modify the definition of an absolute operator  $\Lambda_Y : T_{k-proj}Y \rightsquigarrow TT^\mu Y$  as follows. Let  $D \in Der(\mu) = Lie(Aut(\mu))$  and let  $\delta(t) \in Aut(\mu)$  be a one-parameter subgroup corresponding to  $D$ . It determines the vector field  $D_Y = \frac{d}{dt}_0 \delta(t)_Y$  on  $T^\mu Y$ , where we identify homomorphisms of  $\mu$  with the corresponding natural transformations. Finally, we obtain a natural operator  $\Lambda_{D,Y} : T_{k-proj}Y \rightsquigarrow TT^\mu Y$  defined by  $\Lambda_{D,Y}(X) = D_Y$  for any  $k$ -projectable vector field  $X$  on  $Y \in Ob(k - \mathcal{FM})$ .

**PROPOSITION 3.** *Let  $F$  be a product-preserving bundle functor on  $k - \mathcal{FM}$ . Then every absolute operator  $\Lambda_Y : T_{k-proj}Y \rightsquigarrow TFY$  is of the form  $\Lambda_{D,Y}$  for some  $D \in Der(\mu)$ , where  $\mu$  is the sequence of the form (2) corresponding to  $F$ .*

**PROOF.** The flow  $Fl_t^{\Lambda_Y}$  of  $\Lambda_Y \in \mathcal{X}(FY)$  is  $k - \mathcal{FM}$ -invariant and (thus) global, because  $FY$  is a  $k - \mathcal{FM}$ -orbit of any open neighbourhood of  $0 \in A_k^{m_k} \times \dots \times A_0^{m_0} = F((i^{[k]}(\mathbf{R}))^{m_k} \times \dots \times (i^{[0]}(\mathbf{R}))^{m_0})$  for some  $m_k, \dots, m_0$ . Thus  $Fl_t^{\Lambda_Y} : FY \rightarrow FY$  is a natural transformation. Let  $\eta_t \in Aut(\mu)$  correspond to  $Fl_t^{\Lambda_Y}$ . Then  $D = \frac{d}{dt}_0 \eta_t \in Der(\mu)$  and  $\Lambda_{D,Y} = \Lambda_Y$ .  $\square$

**3. Natural affinors on  $T^\mu$  and natural operators  $T_{k-proj}Y \rightsquigarrow TT^\mu$ .** Let  $\mu$  be a sequence of the form (2) and let  $Y$  be a  $k$ -fibered manifold of the form (1).

Let us recall that a natural affinor on  $T^\mu Y$  is a system of  $k - \mathcal{FM}$ -invariant affinors (i.e., tensor fields of type (1,1))

$$L_Y : TT^\mu Y \rightarrow TT^\mu Y$$

on  $T^\mu Y$  for any  $k - \mathcal{FM}$ -object  $Y$ . The  $k - \mathcal{FM}$ -invariance means that for any  $k - \mathcal{FM}$ -map  $f : Y \rightarrow \bar{Y}$ , there is  $L_{\bar{Y}} \circ TT^\mu f = TT^\mu f \circ L_Y$ .

For  $(y_k, y_{k-1}, \dots, y_0) \in T(T^{A_k}Y_0 \times T^{A_{k-1}}Y_1 \times \dots \times T^{A_0}Y_k) \cap TT^\mu Y$  and  $c \in A_k$  we put

$$(5) \quad L(c)_Y(y_k, y_{k-1}, \dots, y_0) = (L(c)_{Y_k}(y_k), L(\mu^k(c))_{Y_{k-1}}(y_{k-1}), \dots, L(\mu^1 \circ \dots \circ \mu^{k-1} \circ \mu^k(c))_{Y_0}(y_0)),$$

where  $L(a)_M : TT^A M \rightarrow TT^A M$  is the Koszul affinor, [7]. We call  $L(c)_Y$  the modified Koszul affinor on  $T^\mu Y$ .

The following theorem characterizes all natural affinors on  $T^\mu Y$ .

**THEOREM 1.** *Let  $\mu$  be a sequence of the form (2) and  $Y \in Ob(k - \mathcal{FM})$  be of the form (1). Then every natural affinor on  $T^\mu Y$  is of the form  $L(c)_Y$  for some  $c \in A_k$ .*

Theorem 1 generalizes the result of [1] for Weil functors on  $\mathcal{M}f$  and the result of Tomáš's [16] for product-preserving bundle functors on  $\mathcal{FM}$  to all product-preserving bundle functors on  $k - \mathcal{FM}$ . A proof of Theorem 1 will follow a proof of Theorem 2.

For a  $k$ -projectable vector field  $X \in \mathcal{X}_{k-proj}(Y)$ , one can define its flow prolongation  $\mathcal{F}X = \frac{d}{dt}_0 F(Fl_t^X) \in \mathcal{X}(FY)$  to a product-preserving bundle functor  $F = T^\mu$  on  $k - \mathcal{FM}$ . (We know that the flow of  $X$  is formed by local  $k - \mathcal{FM}$ -isomorphisms, and then we can apply  $F = T^\mu$  and obtain a flow on  $FY$ .) One can verify the Kolář formula

$$(6) \quad \mathcal{F}X = \eta_Y \circ FX ,$$

where  $\eta_Y : FTY = T^{id \otimes \mu} Y \cong T^{\mu \otimes id} Y = TFY$  is the exchange isomorphism and  $X$  is considered as  $k - \mathcal{FM}$ -map  $X : Y \rightarrow TY = T^{id} Y$ . We will not use this formula.

The following theorem modifies Kolář's result [7] for Weil functors on  $\mathcal{M}f$  and Tomáš's result [16] for product-preserving bundle functors on  $\mathcal{FM}$  to all product-preserving bundle functors on  $k - \mathcal{FM}$ .

**THEOREM 2.** *Let  $F$  be a product-preserving bundle functor on  $k - \mathcal{FM}$ . Further, let  $X$  be a  $k$ -projectable vector field on a  $k$ -fibered manifold  $Y$  of the form (1). Then any natural operator  $\Lambda_Y : T_{k-proj} Y \rightsquigarrow TFY$  is of the form*

$$L(c)_Y \circ \mathcal{F}X + \Lambda_{D,Y}$$

for some  $c \in A_k$  and  $D \in Der(\mu)$ , where  $\mu$  is the sequence of the form (2) associated to  $F$ .

**PROOF OF THEOREM 2.**  $\Lambda_Y(0)$  is an absolute operator. Thus replacing  $\Lambda_Y$  by  $\Lambda_Y - \Lambda_Y(0)$  and applying Proposition 3 we can assume that  $\Lambda_Y(0) = 0$ .

Since any  $k$ -projectable vector field  $X$  on  $Y \in Ob(k - \mathcal{FM})$  covering non-vanishing vector field on  $Y_0$  is  $\frac{\partial}{\partial x}$  on  $i^{[k]}(\mathbf{R}) \subset i^{[k]}(\mathbf{R}) \times \dots$  in some  $k - \mathcal{FM}$ -coordinates (where the dots denote the respective multiproduct of  $i^{[l]}(\mathbf{R})$ 's),  $\Lambda_Y$  is uniquely determined by  $\Lambda_{i^{[k]}(\mathbf{R}) \times \dots}(\rho \frac{\partial}{\partial x}) : A_k \times \dots \rightarrow A_k \times \dots$ ,  $\rho \in \mathbf{R}$ . Using the invariance with respect to the homotheties being  $k - \mathcal{FM}$ -morphisms  $i^{[k]}(\mathbf{R}) \times \dots \rightarrow i^{[k]}(\mathbf{R}) \times \dots$  and the homogeneous function theorem and  $\Lambda_{i^{[k]}(\mathbf{R}) \times \dots}(0) = 0$  we deduce that for any  $\rho$  the map  $\Lambda_{i^{[k]}(\mathbf{R}) \times \dots}(\rho \frac{\partial}{\partial x}) :$

$A_k \times \dots \rightarrow A_k \times \dots$  is constant and linearly dependent on  $\rho$ . Then using the invariance with respect to  $tid_{i^{[k]}(\mathbf{R})} \times id$  we deduce that the map  $\Lambda_{i^{[k]}(\mathbf{R}) \times \dots}(\rho \frac{\partial}{\partial x}) : A_k \times \dots \rightarrow A_k \times \{0\}$  is constant and linearly dependent on  $\rho$ . Then the vector space of all natural operators  $\Lambda_Y$  as above with  $\Lambda_Y(0) = 0$  is at most  $dim_{\mathbf{R}} A_k$ -dimensional. But all natural operators  $L(c)_Y \circ \mathcal{F}$  form a  $dim_{\mathbf{R}} A_k$ -dimensional vector space. Thus the proof is complete.  $\square$

PROOF OF THEOREM 1. The vectors  $\mathcal{T}^\mu X_v$  for  $X \in \mathcal{X}_{k-proj}(Y)$  and  $v \in T^\mu Y$  form a dense subset in  $TT^\mu Y$  for sufficiently high fiber-dimensional  $Y_k, \dots, Y_0$ . (It is a simple consequence the rank theorem implying that for any Weil algebra  $A$  with  $width(A) = k$  the vector  $\mathcal{T}^A \frac{\partial}{\partial x^1} j^A(t^1, \dots, t^k, 0, \dots, 0) = j^{A \otimes \mathbf{D}}(t^1, \dots, t^k, 0, \dots, 0, t)$  has dense  $\mathcal{M}f_m$ -orbit in  $TT^A \mathbf{R}^m = T^{A \otimes \mathbf{D}} \mathbf{R}^m$  if  $m \geq k + 1$ .) Thus a natural affiner  $L_Y$  on  $T^\mu Y$  is determined by  $L_Y \circ \mathcal{T}^\mu X$  for  $X$  as above. But  $\Lambda_Y : X \rightarrow L_Y \circ \mathcal{T}^\mu X$  is a natural operator with  $\Lambda_Y(0) = 0$ . Thus by the proof of Theorem 2 there is  $\Lambda_Y(X) = L(c)_Y \circ \mathcal{T}^\mu X$  for some  $c \in A_k$ . Then  $L_Y = L(c)_Y$ . For arbitrary  $Y$ , we locally decompose  $id_Y$  by  $p \circ j$  for  $k - \mathcal{FM}$ -maps, where  $j : Y \rightarrow \bar{Y}$  with sufficiently high fiber-dimensional  $\bar{Y}$ . Next, we use the equality  $L_{\bar{Y}} = L(c)_{\bar{Y}}$  and the invariance of natural affiners with respect to  $j$ .  $\square$

According to formula (6), it is sufficient to verify it for  $X = \frac{\partial}{\partial x}$ ; see proof of Theorem 2. But then this is simple to verify.

**4. Final remarks.** Let  $m = (m_k, m_{k-1}, \dots, m_0) \in (\mathbf{N} \cup \{0\})^{k+1}$ . A  $k$ -fibered manifold  $Y$  of the form (1) is  $m$ -dimensional if  $dim(Y_0) = m_0$ ,  $dim(Y_1) = m_0 + m_1, \dots, dim(Y_k) = m_0 + m_1 + \dots + m_k$ . All  $k$ -fibered manifolds of dimension  $m = (m_k, \dots, m_0)$  and their local  $k - \mathcal{FM}$ -isomorphisms form a category which we will denote by  $k - \mathcal{FM}_m$ . It is local and admissible in the sense of [8].

Let  $F = T^\mu : k - \mathcal{FM} \rightarrow \mathcal{FM}$  be a product preserving bundle functor and let  $\eta : F|_{k - \mathcal{FM}_m} \rightarrow F|_{k - \mathcal{FM}_m}$  be a  $k - \mathcal{FM}_m$ -natural transformation. Assume that  $m_k, m_{k-1}, \dots, m_0$  are positive integers. Then by a similar method as for Weil bundles on  $\mathcal{M}f$  one can show that there exists one and only one natural transformation  $\tilde{\eta} : F \rightarrow F$  extending  $\eta$ . Thus by Theorem 1, one can obtain the  $k - \mathcal{FM}_m$ -version of Theorem 1.

THEOREM 1'. Let  $\mu$  be a sequence of the form (2) and  $Y \in Ob(k - \mathcal{FM}_m)$  be of the form (1),  $m = (m_k, \dots, m_0)$ ,  $m_k, \dots, m_0$  positive integers. Then every  $k - \mathcal{FM}_m$ -natural affiner on  $T^\mu Y$  is of the form  $L(c)_Y$  for some  $c \in A_k$ .

By a simple modification of the proof of Theorem 2 one can obtain the  $k - \mathcal{FM}_m$ -version of Theorem 2.

**THEOREM 2'.** *Let  $\mu, Y, m$  be as in Theorem 1'. Further, let  $X$  be a  $k$ -projectable vector field on a  $k$ -fibered manifold  $Y$  of the form (1) and dimension  $m$ . Then any  $k - \mathcal{FM}_m$ -natural operator  $\Lambda_Y : T_{k\text{-proj}}Y \rightsquigarrow TT^\mu Y$  is of the form  $L(c)_Y \circ T^\mu X + \Lambda_{D,Y}$  for some  $c \in A_k$  and  $D \in \text{Der}(\mu)$ .*

The authors would now like to announce that in [14] they describe all product preserving bundle functors on the category  $\mathcal{F}^2\mathcal{M}$  of fibered-fibered manifolds (i.e. fibered surjective submersions between fibered manifolds) and in a paper being in preparation they extend Kolář's result [7] to product-preserving bundle functors on  $\mathcal{F}^2\mathcal{M}$ .

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