Connecting orbits for a singular nonautonomous real Ginzburg-Landau type equation

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Abstract

We propose a method for computation of stable and unstable sets associated to hyperbolic equilibria of nonautonomous ODEs and for computation of specific type of connecting orbits in nonautonomous singular ODEs. We apply the method to a certain a singular nonautonomous real Ginzburg-Landau type equation, which that arises from the problem of formation of spots in the Swift-Hohenberg equation.

Keywords: computer-assisted proof, nonautonomous ODE, heteroclinic connection, Swift-Hohenberg equation, unstable manifold, rigorous numerics
AMS subject classification: 34C45, 65G20, 37C29

1 Introduction.

The aim of this paper is to present a method for computation of connecting orbits between hyperbolic equilibria in nonautonomous ODEs. Although the algorithm we propose is general and may be applied to an ODE in any (finite) dimension, the main motivation for undertaking this research was the problem of finding transversal solutions for the following nonautonomous real Ginzburg-Landau type equation

$$A_{rr} + \frac{A_r}{r} - \frac{A}{4r^2} = A - A^3, \tag{1}$$

where $A : [0, \infty) \to \mathbb{R}$ satisfies

$$A(0) = 0, \quad \lim_{r \to \infty} A(r) = 0. \tag{2}$$

This question comes from McCalla and Sandstede work [MS] on spots in the Swift-Hohenberg equation. The existence of transversal solutions (1) satisfying (2) is listed there as an assumption in the construction of localized solutions called the B spots. The work of Scheel [S] (see also references given there) indicates that bounded solutions of problem (1,2) might be also interpreted as rotating wave solutions to reaction diffusion systems in the plane.

Let us list first several known results about (1,2). An infinite number of solutions with any number of zeros, but without transversality, follows from results of Ryder [R]. Namely, it was shown in [MS] (in the appendix) that after substitution $A(r) = \frac{w(r)}{\sqrt{r}}$ (1) becomes $w" = w - \frac{w^3}{r}$ and this equation fits into the framework in the paper by Ryder (see equation (1.7) in [R]). The methods in [R] are variational and can be applied to wider class of equations. The work by by Scheel [S] also contains similar result, but it was pointed out in [MS] that the proof given there contains a gap. Recently, a computer assisted proof of the existence of one transversal solution with one local extremum was given in [BGW].

The following theorem summarizes our main results about the system (1).

Theorem 1 *For $n = 1, \ldots, 6$ there exists a solution $A_n : (0, \infty) \to \mathbb{R}$ to (1,2) such that:*

- $\lim_{r \to \infty}(A_n(r), A'_n(r)) = (0, 0)$,

- $\lim_{r \to 0^+}(A_n(r), A'_n(r)r) = (0, 0)$,

- *The function $r \to A_n(r)$ has exactly n local extremes in the domain $r \in (0, \infty)$.*

The solution A_n is transversal in the following sense. The sets

$$
\begin{aligned}
W^u \;=\; & \{(r_0, A_0, A'_0)) \in (0, \infty) \times \mathbb{R} \times \mathbb{R} \mid \\
& \text{the backward solution of (1) } A(r) \text{ with initial condition} \\
& A(r_0) = A_0,\ A'(r_0) = A'_0 \text{ satisfies } \lim_{r \to 0}(A(r), A'(r)r) = 0\} \\
W^s \;=\; & \{(r_0, A_0, A'_0)) \in (0, \infty) \times \mathbb{R} \times \mathbb{R} \mid \\
& \text{the forward solution of (1) } A(r) \text{ with initial condition} \\
& A(r_0) = A_0,\ A'(r_0) = A'_0 \text{ satisfies } \lim_{r \to \infty}(A(r), A'(r)) = 0\}
\end{aligned}
$$

are two dimensional immersed manifolds in the extended phase space (r, A, A') and they intersect transversally along the solution curve defined by $A_n(r)$, i.e. $(r, A_n(r), A'_n(r))$.

Plots of the six connecting orbits resulting from Theorem 1 are shown in Fig. 1. The solutions differ in the number of local extremes or the number of

sign changes. The orbit for $n = 1$, with one maximum only, is the one already proved in [BGW]. We present a proof of the existence of six such orbits, but without much difficulty it would be possible to validate the existence of several orbits with more local extremes. The question of the existence of an infinite number of geometrically different transversal solutions for (1,2) is still open.

Our approach is based on the shooting method (see for example [HM]), which is a very well known technique in the dynamics of ODEs and boundary value problems for ODEs.

To realize the shooting method for problem (1,2) we have to deal with the following issues

1. the singularity for $r = 0$,

2. the estimation of stable manifold of the point $(A, A') = (0, 0)$ for non-autonomous equation (1).

Issue 1 is dealt with via the change of the independent variable $\rho = \ln r$. Then the equation (1) becomes

$$w" - w/4 = e^{2\rho}(w - w^3). \tag{3}$$

Now we seek a solution such that $\lim_{\rho \to -\infty} w(\rho) = 0$, so we need estimates for the unstable set of $(w = 0, w' = 0)$ for $\rho \to -\infty$ for our non-autonomous ODE (3), which is basically issue 2 above.

Issue 2 in our work is split in two separate problems

- an analytical derivation of bounds for the stable manifold in the neighborhood of the hyperbolic fixed point, which are valid up to some 'macroscopic' distance from it,

- the globalization of the local (un)stable manifolds using the rigorous numerical integrator from CAPD [CAPD].

Obtaining explicit bounds on the local (un)stable manifolds associated to hyperbolic equilibria in nonautonomous systems is an interesting issue itself and most likely this is the main mathematical contribution of this paper. For this end we adopt the method of cone conditions based on quadratic forms from [ZCC, SZ] to the case of nonautonomous systems. The main novelty of this paper in this context is Lemma 4 which gives explicit bounds on the derivative of the parameterization of the stable manifold of a hyperbolic equilibrium with respect to the time variable. The method is general and may be applied to any smooth nonautonomous system in arbitrary finite dimension. We believe it should be an important ingredient when attacking the question of the existence of infinite number of geometrically different solutions to (1,2) which is the next logical step for this project.

To demonstrate the effectiveness of our estimates of local (un)stable manifolds in Sections 4 and 5 we present easy calculations showing that using our approach we can estimate invariant manifolds for the system (1) up to the distance of order 0.1 from the origin. Outside of this region the stable and unstable
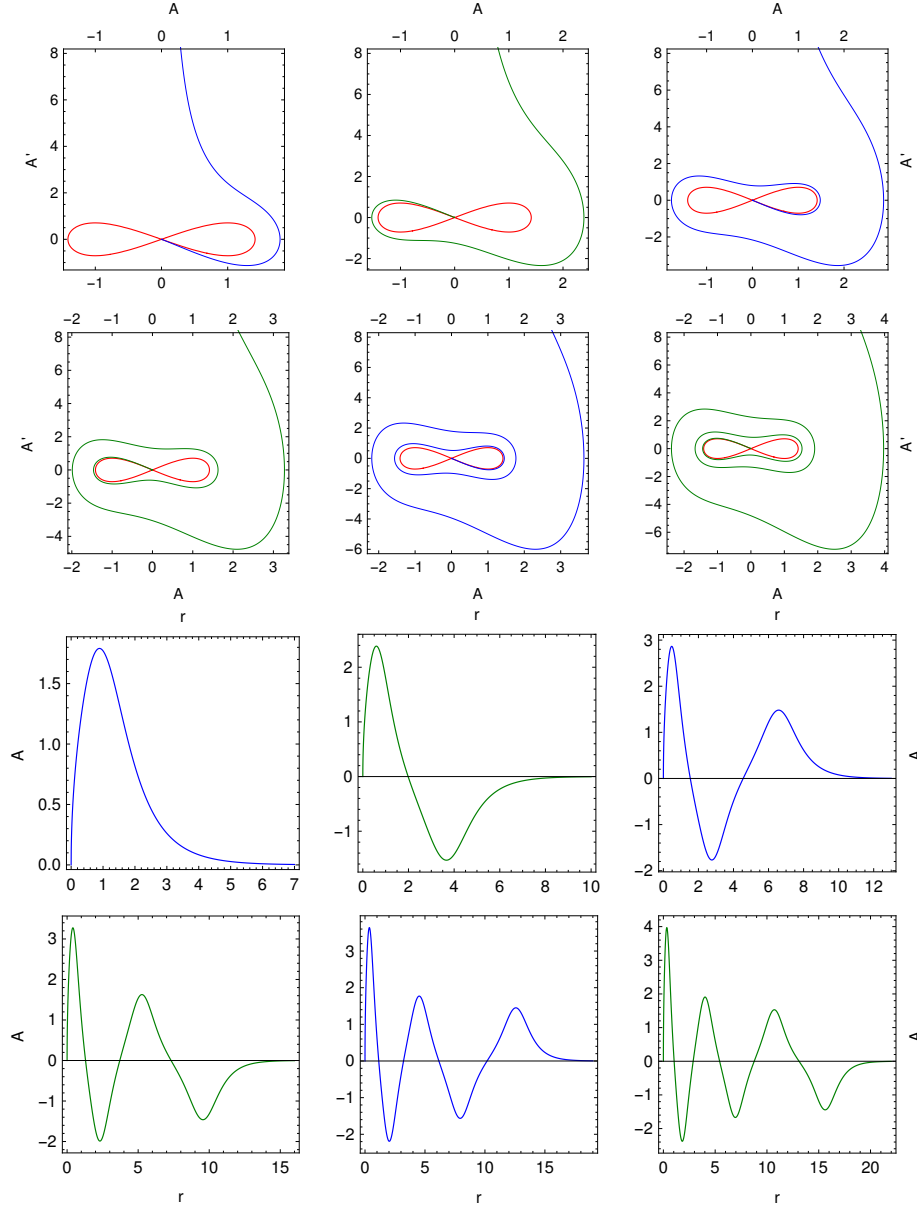
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Figure 1: The six transverse connecting orbits resulting from Theorem 1. In two upper rows in red we have plotted the separatrices (stable and unstable manifolds of $(0,0)$) of system $A_{rr} = A - A^3$, which is the formal limit of (1) for $r \to \infty$.

4

manifolds of (1) and (3) are continued to obtain a connecting orbit using a rigorous ODE integrator from the CAPD library [CAPD], but for this to succeed it is essential that our local bounds are valid up to a 'macroscopic' distance from the origin, otherwise the execution of the rigorous numerics will take very long time and the quality of bounds will be poor. Let us emphasize that to glue unstable and stable manifolds we need to integrate the system and associated variational equations over a finite time interval, only. Our local estimates on the invariant manifolds and bounds coming from the rigorous integration were sharp enough to fulfill the assumptions of the shooting method that guarantees the existence of six connecting orbits as stated in Theorem 1.

Let us comment about the computer assisted proof from [BGW]. There the problem is formulated in the functional analytic language. Differential equation (1) with the help of Green function is replaced by an integral equation on finite domain and condition (2) is encoded in the choice of the function space. Therefore obtaining a desired solution of (1,2) is reduced to the fixed point problem for some nonlinear integral operator in some infinite dimensional function space. This should be contrasted with our approach, where we work in the phase space of our ODE.

The content of this paper can be described as follows. In Section 2 we recall the main geometric tools used in our approach to the (un)stable manifolds. In Section 3 we present theorems which provide us with bounds on the derivatives of (un)stable manifolds. In Sections 4 and 5 we give explicit bounds for (un)stable manifolds and theirs derivatives for (1) and (3). In Section 6 we give a computer assisted proof of Theorem 1.

2 Geometric tools for invariant manifolds of fixed points

To make this paper reasonably self-contained in this section we gather all necessary definitions and geometric theorems from [ZGi, WZ, ZCC, WZ1] related to the rigorous investigation of invariant manifolds of fixed points, with modifications required for the setting of non-autonomous ODEs.

2.1 Notation

For a matrix $A \in \mathbb{R}^{n \times k}$ by A^T we will denote its transpose. Let Q be a square matrix. With some abuse of notation we will denote by the same letter a symmetric matrix Q and associated quadratic form $Q(x) = x^T \cdot Q \cdot x$. To avoid ambiguity we will always use brackets $Q(\cdot)$ to indicate that Q is treat as a quadratic form and Qx or $Q \cdot (x_1 - x_2)$ when we compute matrix-vector product. For a bilinear map $M : V \times W \to \mathbb{R}$ we will use the norm $\|W\| = \max_{\|v\| \leq 1, \|w\| \leq 1} |M(v, w)|$.

For $u \in \mathbb{N}$, $p \in \mathbb{R}^u$ and $r > 0$ by $B_u(p, r)$ we will denote an open ball of radius r centered at p. We will also often use $B_u = B_u(0, 1)$. For a map $h : [0, 1] \times X \to X$ and $t \in [0, 1]$ we set $h_t(x) = h(t, x)$.

For a function $f(t, x)$ we set $D_t f(t, x) = \frac{\partial f}{\partial t}(t, x)$ and $D_x(t, x) = \frac{\partial f}{\partial x}(t, x)$.

2.2 Horizontal and vertical discs

The main tools for proving the existence of local (un)stable manifolds in a given domain are isolating blocks and the cone conditions. The invariant manifolds are then constructed as graphs of smooth functions in some isolating blocks, as horizontal/vertical discs in h-sets. The size of an h-set gives an explicit range in which we can parameterize the local invariant manifold. The cone conditions are used to obtain both the existence and the local uniqueness of the invariant manifold as well as bounds on the derivatives of this parameterization.

Definition 1 *[ZCC, Def. 1]*
   An h-set N is a quadruple $(|N|, u(N), s(N), c_N)$ such that

- *$|N|$ is a compact subset of $\mathbb{R}^n$,*

- *$u(N), s(N) \in \{0, 1, 2, \ldots\}$ are such that $u(N) + s(N) = n$,*

- *$c_N : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$ is a homeomorphism such that $c_N(|N|) = \overline{B}_{u(N)} \times \overline{B}_{s(N)}$.*

We set

$$\begin{aligned}
\dim(N) &:= n, \\
N_c &:= \overline{B}_{u(N)} \times \overline{B}_{s(N)}, \\
N_c^- &:= \partial B_{u(N)} \times \overline{B}_{s(N)}, \\
N_c^+ &:= \overline{B}_{u(N)} \times \partial B_{s(N)}, \\
N^- &:= c_N^{-1}(N_c^-), \\
N^+ &:= c_N^{-1}(N_c^+).
\end{aligned}$$

Hence an *h-set* N is a product of two closed balls in some coordinate system c_N. We call numbers $u(N)$ and $s(N)$ unstable and stable dimensions, respectively. The subscript c refers to the new coordinates given by c_N. The set $|N|$ is called the support of an *h-set*. We often drop bars in the symbol $|N|$ and use N to denote both the *h-set* and its support.

Occasionally we will say that $N = \{(x_0, y_0)\} + \overline{B}_u(0, r_1) \times \overline{B}_s(0, r_2) \subset \mathbb{R}^u \times \mathbb{R}^s$ is an *h-set*. By this we will understand a 'natural' *h-set* structure on N given by: $u(N) = u$, $s(N) = s$, $c_N(x, y) = \left(\frac{x - x_0}{r_1}, \frac{y - y_0}{r_2} \right)$. In the context of $\mathbb{R}^2$ and $u = 1$, $s = 1$ we will sometimes write $N = z_0 + [-a, a] \times [-b, b]$. This is compatible with the above convention as a defines the radius of the ball $\overline{B}_u(0, a) = [-a, a]$ and b of $\overline{B}_s(0, b) = [-b, b]$.

Definition 2 *[ZCC, Def. 5]*
   Let N be an h-set. Let $b : \overline{B}_{u(N)} \to |N|$ be a continuous mapping and let $b_c = c_N \circ b$. We say that b is a horizontal disc *in N if there exists a homotopy*
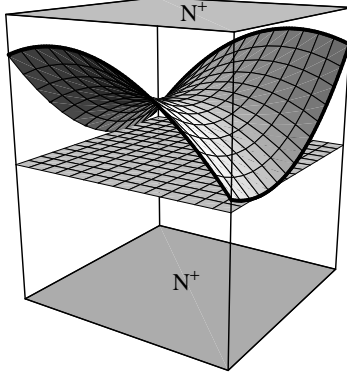
Figure 2: An h-set N with $u = 2$, $s = 1$ and a horizontal disc in N.

$h : [0,1] \times \overline{B}_{u(N)} \to N_c$, such that

$$
\begin{aligned}
h_0 &= b_c, \\
h_1(x) &= (x,0), \quad \forall x \in \overline{u(N)}, \\
h(t,x) \in N_c^-, &\qquad \forall t \in [0,1] \text{ and } \forall x \in \partial B_{u(N)}.
\end{aligned}
$$

The geometry of this definition is shown in Fig. 2. During the homotopy h deforming a horizontal disc to the map $h(1,x) = (x,0)$ the image of the boundary $h([0,1] \times B_{s(N)})$ must remain in N_c^-. In a similar way we can define vertical discs.

Definition 3 *[ZCC, Def. 6] Let N be an h-set. Let $b : \overline{B}_{s(N)} \to |N|$ be a continuous mapping and let $b_c = c_N \circ b$. We say that b is a vertical disc in N if there exists a homotopy $h : [0,1] \times \overline{B}_{s(N)} \to N_c$, such that*

$$
\begin{aligned}
h_0 &= b_c, \\
h_1(x) &= (0,x), \quad \forall x \in \overline{B}_{s(N)}, \\
h(t,x) \in N_c^+, &\qquad \forall t \in [0,1] \text{ and } \forall x \in \partial B_{s(N)}.
\end{aligned}
$$

Definition 4 *[ZCC, Def. 7] Let N be an h-set in $\mathbb{R}^n$ and b be a horizontal (vertical) disc in N. We will say that $x \in \mathbb{R}^n$ belongs to b, when $b(z) = x$ for some $z \in \mathrm{dom}(b)$.*

By $|b|$ we will denote the image of b. Hence $z \in |b|$ iff z belongs to b.

2.3 Cone conditions and the stable manifold theorem

Below we present definitions and theorems that allow us to handle and verify hyperbolic structures for nonautonomous ODEs using h-sets and quadratic forms.
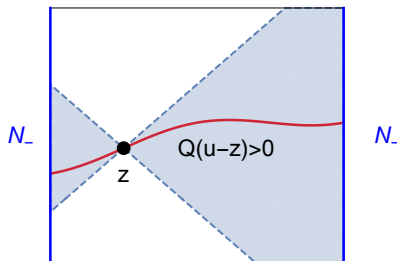
Figure 3: An h-set N and a horizontal disc in N satisfying the cone condition.

Definition 5 *[ZCC, Def. 8] Let $N \subset \mathbb{R}^n$ be an h-set and $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form, such that*

$$Q(x,y) = \alpha(x) - \beta(y), \quad (x,y) \in \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)},$$

where $\alpha : \mathbb{R}^{u(N)} \to \mathbb{R}$ and $\beta : \mathbb{R}^{s(N)} \to \mathbb{R}$ are positive definite quadratic forms.
The pair (N, Q) is called an h-set with cones.

We will often omit Q in the symbol (N, Q) and will say that N is an *h-set with cones*.

Definition 6 *[ZCC, Def. 9] Let (N, Q) be an h-set with cones and $b : \overline{B}_u \to |N|$ be a horizontal disc.*
We will say that b satisfies the cone condition (with respect to Q) iff for any $x_1, x_2 \in \overline{B}_u$, $x_1 \neq x_2$ the following inequality holds:

$$Q(b_c(x_1) - b_c(x_2)) > 0.$$

The geometry of this definition is shown in Fig. 3. The entire graph of horizontal disc b must lie in the positive cone attached to any point z that belongs to the horizontal disc b (except point z). It can be shown [ZCC] that a horizontal disc satisfying the cone condition must be of the form $b_c(x, y) = (x, g(x))$ where g is a Lipschitz function with an explicit bound on Lipschitz constant coming from the quadratic form. We will use this property to control derivative of the parameterization of the local invariant manifold which will be constructed as a horizontal (vertical) disc in some h-set satisfying the cone condition.

Definition 7 *[ZCC, Def. 10]*
Let (N, Q) be an h-set with cones and $b : \overline{B}_s \to |N|$ be a vertical disc.
We will say that b satisfies the cone condition (with respect to Q) iff for any $y_1, y_2 \in \overline{B}_s$, $y_1 \neq y_2$ the following inequality holds:
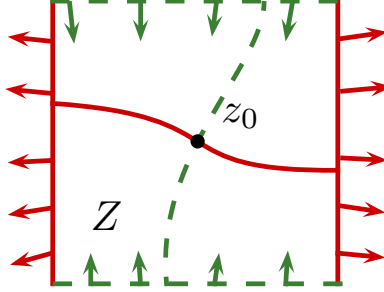
$$Q(b_c(y_1) - b_c(y_2)) < 0.$$

Figure 4: An isolating block Z for some planar ODE. The stable (vertical green dashed line) and unstable (horizontal red solid line) manifolds for φ inside Z are plotted, arrows indicate the vector field f. Dashed green and solid red border lines indicate the δ-sections Σ^+ and Σ^- respectively.

Let us consider an nonautonomous ODE

$$z' = f(t, z), \tag{4}$$

where $z \in \mathbb{R}^n$, $f \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. Let us denote by $\varphi(t, t_0, p)$ the solution of (4) with the initial condition $z(t_0) = p$.

Below we present the definitions of the local stable and unstable set for a fixed point of an non-autonomous of ODE, which are just an adaptation of the analogous notions for autonomous ODEs.

Definition 8 *Let $z_0 \in \mathbb{R}^n$ be such that $f(t, z_0) = 0$ for $t \in \mathbb{R}$. We will call such z_0 a fixed point for (4) (or f).*

Let $Z \subset \mathbb{R}^n$, $z_0 \in Z$ be a fixed point for (4) and let $t_0 \in \mathbb{R}$. We define

$$W^s_{t_0,Z}(z_0, \varphi) = W^s_{t_0,Z}(z_0, f) = \left\{ z : \forall_{t \geq 0} \varphi(t + t_0, t_0, z) \in Z, \ \lim_{t \to \infty} \varphi(t, t_0, z) = z_0 \right\},$$

$$W^u_{t_0,Z}(z_0, \varphi) = W^u_{t_0,Z}(z_0, f) = \left\{ z : \forall_{t \leq 0} \varphi(t + t_0, t_0, z) \in Z, \ \lim_{t \to -\infty} \varphi(t, t_0, z) = z_0 \right\}.$$

When φ, f or z_0 is known from the context, then we will often drop it and write $W^{u,s}_{t_0,Z}(z_0)$ or $W^{u,s}_{t_0,Z}$.

Below we recall the notion of the isolating block from the Conley index theory with some obvious modification for nonautonomous ODEs.

Definition 9 *Let φ be an autonomous local flow induced by $f : \mathbb{R}^n \to \mathbb{R}^n$. For $\delta > 0$ the set $\Sigma \subset \mathbb{R}^n$ is called a $\delta-$section for the flow φ iff $\varphi((-\delta, \delta), \Sigma)$ is an open set and the map $\sigma : \Sigma \times (-\delta, \delta) \to \varphi((-\delta, \delta), \Sigma)$ defined by $\sigma(x, t) = \varphi(t, x)$ is a homeomorphism.*

9

For a nonautonomous ODE $x' = f(t, x)$ inducing φ we say that Σ as above is δ-section for φ for $t \in T \subset \mathbb{R}$, iff for each $t_0 \in T$ the set Σ is a δ section for the vector field $x \to f(t_0, x)$ (i.e. we freeze the time).

Let $B \subset \mathbb{R}^n$ be a compact set and $T \subset \mathbb{R}$. B is called an isolating block for non-autonomous ODE for $t \in T$ iff $\partial B = B^- \cup B^+$, where B^- and B^+ are closed sets, and there exists $\delta > 0$ and two $\delta-$sections, Σ^+ and Σ^- such that

$$B^+ \subset \Sigma^+, \quad B^- \subset \Sigma^-,$$
$$\forall x \in B^+ \quad \forall t_0 \in T \quad \forall t \in (-\delta, 0) \qquad \varphi(t + t_0, t_0, x) \notin B,$$
$$\forall x \in B^- \quad \forall t_0 \in T \quad \forall t \in (0, \delta) \qquad \varphi(t, x) \notin B.$$

B^- and B^+ will called an exit set and an entrance set, respectively.

In the present paper we will use *h-sets* which are isolating blocks for $t \geq t_0$ (or for $t \leq t_0$). Simply, it means that N^+ and N^- are sections of the vector field.

Definition 10 *Let N be an h-set in $\mathbb{R}^n$. We say that N is an isolating block for nonautonomous ODE (4) for $t \geq t_0$, iff N^- and N^+ are δ-sections for $f(t, \cdot)$ for $t \geq t_0$ as in Definition 9.*

Therefore, if N, h-set, is an isolating block, then N^- and N^+ are the exit and the entrance sets, respectively.

Definition 11 *Let N be an h-set, such that c_N is a diffeomorphism. For a vector field f on $|N|$ we define a vector field on N_c by*

$$f_c(z) = Dc_N(c_N^{-1}(z))f(c_N^{-1}(z)). \tag{5}$$

Observe that f_c is the vector field f expressed in the new variables. If f depends on time, then f_c is time dependent too.

Definition 12 *[ZCC, Def. 13]*
Let $U \subset \mathbb{R}^n$ be such that $U = \overline{U}$ and $\text{int } U \neq \emptyset$. Let $g : U \to \mathbb{R}^m$ be a C^1 function. We define the interval enclosure of $Dg(U)$ by

$$[Dg(U)] := \left\{ A \in \mathbb{R}^{n \times m} : \forall_{i,j} A_{ij} \in \left[\inf_{x \in U} \frac{\partial g_i(x)}{\partial x_j}, \sup_{x \in U} \frac{\partial g_i(x)}{\partial x_j} \right] \right\}.$$

We say that an interval matrix $[A] \subset \mathbb{R}^{n \times n}$ (i.e. a set of matrices) is positive definite if for all symmetric matrices $A \in [A]$ A is positive definite.

The lemma below serves to motivate our definition of the cone condition for ODEs.

Lemma 2 *Assume that Q is a quadratic form on $\mathbb{R}^n$.*
Consider equation (4). Let $I \subset \mathbb{R}$ be an interval. Assume that $U \subset \mathbb{R}^n$ is convex and the following condition is satisfied for $t \in I$

$$matrix \quad [D_z f(t, U)]^T Q + Q[D_z f(t, U)] \quad is \ positive \ definite. \tag{6}$$

Let $z_1(t)$ and $z_2(t)$ be two different solutions of (4), such that for $t \in I$ $z_i(t) \in U$.
Then there exists $\eta > 0$ such that

1.

$$\frac{d}{dt}Q(z_1(t) - z_2(t)) \quad > \quad 0, \quad t \in I, \tag{7}$$

$$\frac{d}{dt}Q(z_1(t) - z_2(t)) \quad > \quad \pm\eta Q(z_1(t) - z_2(t)), \quad t \in I \tag{8}$$

2. if $t_0, t + t_0 \in I$, $t > 0$ and $Q(z_1(t_0) - z_2(t_0)) > 0$, then

$$Q(z_1(t_0 + t) - z_2(t_0 + t)) \geq e^{\eta t}Q(z_1(t_0) - z_2(t_0)), \tag{9}$$

3. if $t_0, t + t_0 \in I$, $t < 0$ and $Q(z_1(t_0) - z_2(t_0)) < 0$, then

$$Q(z_1(t_0 + t) - z_2(t_0 + t)) \leq e^{\eta|t|}Q(z_1(t_0) - z_2(t_0)). \tag{10}$$

Proof: We have

$$
\begin{aligned}
\frac{d}{dt}Q(z_1(t) - z_2(t)) &= (f(t, z_1(t)) - f(t, z_2(t)))^T \cdot Q \cdot (z_1(t) - z_2(t)) \\
&+ (z_1(t) - z_2(t))^T \cdot Q \cdot (f(t, z_1(t)) - f(t, z_2(t))) \\
&= (z_1(t) - z_2(t))^T \cdot C^T \cdot Q \cdot (z_1(t) - z_2(t)) \\
&+ (z_1(t) - z_2(t))^T \cdot Q \cdot C(z_1(t) - z_2(t)) \\
&= (z_1(t) - z_2(t))^T \cdot (C^T Q + QC) \cdot (z_1(t) - z_2(t)),
\end{aligned}
$$

where

$$C = C(t, z_1, z_2) = \int_0^1 D_z f(t, z_1 + s(z_2 - z_1))ds.$$

Observe that $C \in [D_z f(t, U)]^T Q + Q[D_z f(t, U)]$, so (7,8) follow easily.

From (8) and differential inequalities we obtain

$$
\begin{aligned}
Q(z_1(t_0 + t) - z_2(t_0 + t)) &\geq e^{\pm\eta t}Q(z_1(t_0) - z_2(t_0)), \quad t > 0, \ t_0, t_0 + t \in I \\
Q(z_1(t_0 + t) - z_2(t_0 + t)) &\leq e^{\pm\eta t}Q(z_1(t_0) - z_2(t_0)), \quad t < 0, \ t_0, t_0 + t \in I
\end{aligned}
$$

From the above conditions one can easily infer the remaining assertions. ∎

Let us remark that from the above lemma it follows that cone condition (6) implies that cones $\mathcal{C}^+(z) = \{u \mid Q(z - u) > 0\}$ are forward invariant for (4) and there is an exponential expansion in $\mathcal{C}^+(z)$. Analogously cones $\mathcal{C}^-(z) = \{u \mid Q(z - u) > 0\}$ are backward invariant and there is an exponential expansion in $\mathcal{C}^-(z)$ in backward time.

The following theorem can be easily obtained using the method as in the proof of Theorem 26 in [ZCC].

Theorem 3 *Consider (4).*
Assume that $z_0 \in \mathbb{R}^n$ is such that for any t holds $f(t, z_0) = 0$.

Assume that (N, Q) is an h-set with cones, such that $z_0 \in \text{int } N$ and which is an isolating block for (4) for $t \geq t_0$, c_N is a diffeomorphism and that the following cone condition is satisfied for $t \geq t_0$:

$$matrix \quad [D_z f_c(t, N_c)]^T Q + Q[D_z f_c(t, N_c)] \quad is \; positive \; definite. \tag{11}$$

Then for any $t_1 \geq t_0$ the set $W^s_{t_1,N}(z_0, f)$ is a vertical disc in N satisfying the cone condition. Therefore there exists a function $x_s : \{t \geq t_0\} \times \overline{B}_s \to \overline{B}_u$, such that

$$W^s_{t_1,N}(z_0, f) = \{c_N^{-1}(x_s(t_1, y), y)) \mid y \in \overline{B}_s\}. \tag{12}$$

Analogous result is valid for the local unstable manifold.

Observe that from the standard results about exponential dichotomies (see for example [C]) it follows that function $x_s(t_1, y)$ is smooth with respect to t_1 and y if f and our coordinate change c_N are smooth.

3 The estimate for the derivative with respect to time of $W^{u,s}_{t,N}(z_0)$ for non-autonomous ODE

In the sequel we represent points in $\mathbb{R}^n$ by $z = (x, y)$ with $x \in \mathbb{R}^u$ and $y \in \mathbb{R}^s$, $n = u + s$.

3.1 The estimate for the time derivative of $W^{u,s}_{t,N}(z_0)$

The following lemma gives computable bounds on the first derivatives of stable manifolds of a hyperbolic fixed point for a nonautonomous ODE with respect to the initial time. The proof is analogous to the proofs of the dependence on parameters of the stable manifolds of maps from [ZCC, Thm. 21] with some refined estimates in [WZ1, Th. 4.1] and for parameterized ODEs [SZ, Th. 5]. The theorem below and its proof is an adaptation of these results for nonautonomous ODEs.

Lemma 4 *The same assumptions as in Theorem 3. N has a natural h-set structure, $N = \bar{B}_u(p_x, r_x) \times \bar{B}_s(p_y, r_y)$ and that the quadratic form is given by (in natural coordinates) $Q(x, y) = a\|x\|^2 - b\|y\|^2$, where $a, b > 0$.*

From the assumption on the cone condition for Q there exists $E > 0$ such that for $t \geq t_0$ and $z \in N$ there holds

$$v^T \left(\left(\frac{\partial f}{\partial z}(t, z) \right)^T Q + Q \frac{\partial f}{\partial z}(t, z) \right) v \geq E\|v\|^2, \quad v \in \mathbb{R}^n. \tag{13}$$

Put

$$m = \sup_{t \geq t_0, z \in N} \left\| \left(\frac{\partial f}{\partial t}(t, z) \right)^T Q + Q \frac{\partial f}{\partial t}(t, z) \right\| \tag{14}$$

12

and

$$\delta = \frac{am^2}{E^2}. \tag{15}$$

Under above assumptions, if $(x_i, y_i) \in W^s_{t_i,N}(z_0)$ and $t_i \geq t_0$ for $i = 1, 2$, then

$$a\|x_1 - x_2\|^2 \leq \delta(t_1 - t_2)^2 + b\|y_1 - y_2\|^2. \tag{16}$$

In particular, if $W^s_{t,N}(z_0) = \{(x_s(t, y), y) \mid y \in \overline{B}_s(p_y, r_y)\}$, then for $t_1, t_2 \geq t_0$ there holds

$$\|x_s(t_1, y) - x_s(t_2, y)\| \leq \frac{m}{E}|t_1 - t_2|. \tag{17}$$

Proof: Consider the quadratic form on the extended phase-space (t, x, y)

$$\widetilde{Q}(t, z) = Q(z) - \delta t^2, \quad \delta > 0.$$

The value of δ will be fixed later during the proof (and it will turn out that (15) is a good guess).

Observe that, if for $i = 1, 2$ $t_i \geq t_0$, $\bar{z}_i \in W^s_{t_i,N}(f)$ and $z_i(t_i + t)$ for $t > 0$ is a solution of (4) such that $z_i(t_i) = \bar{z}_i$, then

$$|\widetilde{Q}((t_1 + t, z_1(t_1 + t)) - (t_2 + t, z_2(t_2 + t)))| \leq L, \quad \text{for } t \in \mathbb{R}_+. \tag{18}$$

Indeed, since $z_i(t_i + t) \in N$ for $t > 0$, so $|Q(z_1(t_1 + t) - z_2(t_2 + t))|$ is bounded, and $\delta((t_1 + t) - (t_2 + t))^2 = \delta(t_1 - t_2)^2$ is constant.

We want find a positive δ, such that the positive cone defined in terms of $\widetilde{Q}$ to be forward invariant for $t \geq t_0$ and we have an exponential growth in this cone, i.e. there exists $c > 0$, such that if $\bar{z}_1, \bar{z}_2 \in N$ and $t_1 \geq t_0$, $t_2 \geq t_0$ are such that $\widetilde{Q}((t_1, \bar{z}_1) - (t_2, \bar{z}_2)) > 0$ and, $z_1(t + t_1)$ and $z_2(t + t_2)$ are solutions of (4) with $z_i(t_i) = \bar{z}_i$, $i = 1, 2$, then for $t > 0$ holds

$$\widetilde{Q}((t_1 + t, z_1(t_1 + t)) - (t_2 + t, z_2(t_2 + t))) > e^{ct}\widetilde{Q}((t_1, \bar{z}_1) - (t_2, \bar{z}_2)) \tag{19}$$

if $z_1(t_1 + s), z_2(t_2 + s) \in N$ for $s \in [0, t]$.

This together with (18) implies that if $\bar{z}_i \in W^s_{t_i,N}(f)$ and $t_i \geq t_0$ for $i = 1, 2$, then

$$\widetilde{Q}((t_1, \bar{z}_1) - (t_2, \bar{z}_2)) \leq 0. \tag{20}$$

From this condition and the computed expression for δ we will obtain our assertions.

So now we turn to finding δ.

Let $z_1(t + t_1)$ and $z_2(t + t_2)$ be as in condition (19). We set

$$L(t) := \widetilde{Q}((t + t_1, z_1(t + t_1)) - (t + t_2, z_1(t + t_2))) =$$
$$Q(z_1(t + t_1) - z_2(t + t_2)) - \delta(t_1 - t_2)^2.$$

We have

$$L'(0) = \frac{d}{dt} Q(z_1(t + t_1) - z_2(t + t_2))_{t=0} =$$
$$(f(t_1, z_1(t_1)) - f(t_2, z_2(t_2)))^T \cdot Q \cdot (z_1(t_1) - z_2(t_2)) +$$
$$(z_1(t_1) - z_2(t_2))^T \cdot Q \cdot (f(t_1, z_1(t_1)) - f(t_2, z_2(t_2))).$$

Observe that

$$f(t_1, z_1(t_1)) - f(t_2, z_2(t_2)) =$$
$$\int_0^1 \frac{\partial f}{\partial t}((1 - s)(t_1, z_1(t_1)) + s(t_2, z_2(t_2)))ds \cdot (t_1 - t_2) +$$
$$\int_0^1 \frac{\partial f}{\partial z}((1 - s)(t_1, z_1(t_1)) + s(t_2, z_2(t_2)))ds \cdot (z_1(t_1) - z_2(t_2)) =$$
$$C_t(t_1 - t_2) + C_z(z_1(t_1) - z_2(t_2)),$$

where

$$C_t = C_t(t_1, z_1, t_2, z_2) = \int_0^1 \frac{\partial f}{\partial t}((1 - s)(t_1, z_1) + s(t_2, z_2))ds,$$

$$C_z = C_z(t_1, z_1, t_2, z_2) = \int_0^1 \frac{\partial f}{\partial z}((1 - s)(t_1, z_1) + s(t_2, z_2))ds.$$

From (13) we immediately obtain

$$v^T \left(C_z^T Q + Q C_z \right) v \geq E\|v\|^2, \quad v \in \mathbb{R}^n.$$

From (14) it follows that

$$m \geq \left\| C_t^T Q + Q C_t \right\|.$$

Continuing the previous derivation we obtain

$$L'(0) = (z_1 - z_2)^T \left(C_z^T Q + Q C_z \right) (z_1 - z_2) +$$
$$(t_1 - t_2)(C_t^T Q)(z_1 - z_2) + (z_1 - z_2)^T (Q C_t)(t_1 - t_2).$$

Therefore we arrived at the following estimate

$$L'(0) = \frac{d}{dt} \widetilde{Q}((t + t_1, z_1(t + t_1)) - (t + t_2, z_2(t + t_2)))_{t=0} \geq$$
$$E\|z_1 - z_2\|^2 - m|t_1 - t_2| \cdot \|z_1 - z_2\|.$$

Now we make the use of the assumption $\widetilde{Q}((t_1, z_1) - (t_2, z_2)) \geq 0$. We have

$$\sqrt{\frac{a}{\delta}} \|z_1 - z_2\| \geq \sqrt{\frac{a}{\delta}} \|x_1 - x_2\| \geq |t_1 - t_2|.$$

14

Therefore we obtain

$$L'(0) \geq E\|z_1 - z_2\|^2 - m\sqrt{\frac{a}{\delta}}\|z_1 - z_2\|^2 = \left(E - m\sqrt{\frac{a}{\delta}}\right)\|z_1 - z_2\|^2.$$

Hence the positive cone for $\widetilde{Q}$ will be forward invariant provided

$$E > m\sqrt{\frac{a}{\delta}},$$

which implies that

$$\delta > \frac{am^2}{E^2}. \tag{21}$$

Let us fix any δ satisfying (21).

From the above reasoning we know that the positive cone for $\widetilde{Q}$ will be forward invariant and if $\widetilde{Q}((t_1, z_1) - (t_2, z_2)) \geq 0$, then for any t, such that $z_i(t_i + t) \in N$ holds

$$\frac{d}{dt}\widetilde{Q}((t + t_1, z_1(t + t_1)) \quad - \quad (t + t_2, z_2(t + t_2))) \geq \left(E - m\sqrt{\frac{a}{\delta}}\right)\|z_1 - z_2\|^2 \geq$$
$$c\widetilde{Q}((t_1 + t, z_1(t_1 + t)) - (t_2 + t, z_2(t_2 + t))$$

for some positive c. Therefore

$$\widetilde{Q}((t + t_1, z_1(t + t_1)) - (t + t_2, z_2(t + t_2))) \geq \widetilde{Q}((t_1, z_1) - (t_2, z_2))e^{ct}$$

for $t > 0$ such that both $z_1(s + t_1)$ and $z_2(s + t_2)$ are in N for $s \in [0, t]$.

Therefore we have established (20) for any δ satisfying inequality (21), but passing to $\delta \to \frac{am^2}{E^2}$ we obtain it also for $\delta = \frac{am^2}{E^2}$.

If we set $\bar{z}_i = (x_i, y_i)$, then from (20) we have the following inequality

$$a\|x_1 - x_2\|^2 \leq \delta(t_1 - t_2)^2 + b\|y_1 - y_2\|^2.$$

This proves (16). Inequality (17) is obtained, when we apply (16) to $\bar{z}_i = (t_i, x_s(t_i, y), y)$.

∎

In the application of Lemma 4 in this paper we will only use estimate (17). Moreover, since we are using the natural coordinates on N cone condition (11) becomes

$$matrix \quad [D_z f(t, N)]^T Q + Q[D_z f(t, N)] \quad is \ positive \ definite. \tag{22}$$

This simply means that we do not need to use the coordinate change c_N.

3.2 Estimate for time derivative of $W_{t,N}^u(z_0)$

By reversing the direction of time from Lemma 4 we obtain the following statement about the derivatives of the unstable manifold.

Lemma 5 *Assume that $f(t, z_0) = 0$ for all $t \in \mathbb{R}$. Let N has a natural h-set structure, $N = \bar{B}_u(p_x, r_x) \times \bar{B}_s(p_y, r_y)$, $z_0 \in \operatorname{int} N$ and that quadratic form Q on is given by (in natural coordinates) $Q(x, y) = a\|x\|^2 - b\|y\|^2$, where $a, b > 0$.*

Assume that for $t \le t_0$ the set N is an isolating block for (4) and f satisfies cone condition (11) on N with respect to quadratic form Q.

From the cone condition for Q there exists $E > 0$ such that for $t \le t_0$ and $z \in N$

$$v^T \left(\left(\frac{\partial f}{\partial z}(t, z) \right)^T Q + Q \frac{\partial f}{\partial z}(t, z) \right) v \ge E\|v\|^2, \quad v \in \mathbb{R}^n.$$

Put

$$m = \sup_{t \le t_0, z \in N} \left\| \left(\frac{\partial f}{\partial t}(t, z) \right)^T Q + Q \frac{\partial f}{\partial t}(t, z) \right\|.$$

Under above assumptions, if $W_{t,N}^u(z_0) = \{(x, y_u(t, x)) \mid x \in \bar{B}_u(0, 1)\}$, then for $t_1, t_2 \le t_0$ holds

$$\|y_u(t_1, x) - y_u(t_2, x)\| \le \frac{m}{E} |t_1 - t_2|.$$

Proof: Transformation $t \to -t$ changes $Q \to -Q$ and $(x, y) \to (y, x)$. The quantities in Lemma 4 are not affected by this change. ∎

4 Estimates for the stable manifold for $r \to \infty$ for equation (1)

In the limit $r \to \infty$ equation (1) becomes

$$u'' = u - u^3. \tag{23}$$

We consider the first order system corresponding to (23).

$$u' = v, \quad v' = u - u^3. \tag{24}$$

It is easy to see that (24) is Hamiltonian with $H(v, u) = \frac{v^2}{2} - \frac{u^2}{2} + \frac{u^4}{4}$.

It is clear that $(0, 0)$ is a hyperbolic fixed for (24). Its stable and unstable manifolds coincide to form an eight-shaped loop (called a separatrix) shown in two upper row drawings in Fig. 1. Any solution of (1) which converges to 0 for $r \to \infty$ when plotted on (A, A') plane should approach this separatrix. In Fig. 1 this is shown for six orbits.

16

We introduce coordinates, which diagonalize the linear part of (24) by

$$\begin{cases} x_{\mathrm{e}} = u + v, \\ y_{\mathrm{e}} = u - v. \end{cases} \tag{25}$$

The inverse is given by

$$\begin{cases} u = (x_{\mathrm{e}} + y_{\mathrm{e}})/2, \\ v = (x_{\mathrm{e}} - y_{\mathrm{e}})/2. \end{cases}$$

In new coordinates system (24) becomes

$$\begin{cases} x'_{\mathrm{e}} = x_{\mathrm{e}} + g(x_{\mathrm{e}}, y_{\mathrm{e}}), \\ y'_{\mathrm{e}} = -y_{\mathrm{e}} - g(x_{\mathrm{e}}, y_{\mathrm{e}}) \end{cases} \tag{26}$$

where

$$g(x, y) = -\frac{x - y}{2r} + \frac{x + y}{8r^2} - \frac{1}{8}(x + y)^3.$$

4.1 Isolating block

We look for the isolating block $N^{\mathrm{e}} = [-d_1^{\mathrm{e}}, d_1^{\mathrm{e}}] \times [-d_2^{\mathrm{e}}, d_2^{\mathrm{e}}]$ for (26).

Lemma 6 *Put $d = d_1^{\mathrm{e}} + d_2^{\mathrm{e}}$ and assume that*

$$\min\{d_1^{\mathrm{e}}, d_2^{\mathrm{e}}\} > d \left(\frac{1}{2r} + \frac{1}{8r^2} + \frac{d^2}{8} \right). \tag{27}$$

Then N^{e} is an isolating block for (26).

Proof:

First we verify the exit condition. For $|x_{\mathrm{e}}| = d_1^{\mathrm{e}}$ we have

$$x'_{\mathrm{e}} x_{\mathrm{e}} = x_{\mathrm{e}}^2 + x_{\mathrm{e}} g(x_{\mathrm{e}}, y_{\mathrm{e}}) =$$

$$x_{\mathrm{e}}^2 - \frac{1}{2} \frac{x_{\mathrm{e}} - y_{\mathrm{e}}}{r} x_{\mathrm{e}} + \frac{(x_{\mathrm{e}} + y_{\mathrm{e}}) x_{\mathrm{e}}}{8r^2} - \frac{(x_{\mathrm{e}} + y_{\mathrm{e}})^3 x_{\mathrm{e}}}{8} \geq$$

$$(d_1^{\mathrm{e}})^2 - \frac{1}{2} \frac{d_1^{\mathrm{e}} d}{r} - \frac{d_1^{\mathrm{e}} d}{8r^2} - \frac{d^3 d_1^{\mathrm{e}}}{8} =$$

$$d_1^{\mathrm{e}} \left(d_1^{\mathrm{e}} - d \left(\frac{1}{2r} + \frac{1}{8r^2} + \frac{d^2}{8} \right) \right).$$

Therefore $x_{\mathrm{e}} x'_{\mathrm{e}} > 0$ when (27) is satisfied.

In a similar way we verify the entry condition. For $|y_{\mathrm{e}}| = d_2^{\mathrm{e}}$ we have

$$- (y'_{\mathrm{e}} y_{\mathrm{e}}) \quad = \quad y_{\mathrm{e}}^2 + y_{\mathrm{e}} g(x_{\mathrm{e}}, y_{\mathrm{e}}) \quad \geq \quad d_2^{\mathrm{e}} \left(d_2^{\mathrm{e}} - d \left(\frac{1}{2r} + \frac{1}{8r^2} + \frac{d^2}{8} \right) \right).$$

Therefore $-(y_{\mathrm{e}} y_{\mathrm{e}})' > 0$ when (27) is satisfied. ∎

4.2 Cone condition

Our cones are defined in terms of the quadratic form

$$Q = \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix}. \tag{28}$$

Let

$$C(N, f) = [\overline{Df}^T(N)]Q + Q[\overline{Df}(N)].$$

where $[\overline{Df}(N)]$ is defined as follows: $M \in [\overline{Df}(N)]$ iff there exists a pair of points $z_1, z_2 \in N$ (can be equal) such that $M = \int_0^1 Df(z_1 + t(z_2 - z_1))dt$.

Lemma 7 *Assume that N is a convex and compact set and $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 smooth. Assume that for every $z \in N$ there exists $\epsilon(z) \geq \epsilon_N > 0$, such that for all $v \in \mathbb{R}^n$ there holds*

$$v^T(Df^T(z)Q + QDf(z))v \geq \epsilon(z)\|v\|^2.$$

Then for every $M \in C(N, f)$ and $v \in \mathbb{R}^n$ there holds

$$v^T(M^TQ + QM)v \geq \epsilon_N\|v\|^2.$$

Proof: For $z_1, z_2 \in N$ and $v \in \mathbb{R}^n$ we have

$$v^T\left(\left(\int_0^1 Df^T(z_1 + t(z_2 - z_1))tdt\right)Q + Q\left(\int_0^1 Df(z_1 + t(z_2 - z_1))dt\right)\right)v =$$

$$\int_0^1 v^T(Df^T(z_1 + t(z_2 - z_1))Q + QDf(z_1 + t(z_2 - z_1)))vdt \geq \epsilon_N\|v\|^2.$$

$\blacksquare$

Lemma 8 *Let $N^e = [-d_1^e, d_1^e] \times [-d_2^e, d_2^e]$ and Q be given by (28) with $a = 1$. Assume that for some $E > 0$ and $r_* > 0$ there holds*

$$1 > \frac{1}{2r_*} + \frac{3}{4}(d_1^e + d_2^e)^2 + E/2.$$

Then for all $v \in \mathbb{R}^n$, $M \in C(N^e, f)$ and $r \geq r_*$

$$v^T(MQ + QM)v \geq E\|v\|^2, \tag{29}$$

i.e. the cone condition holds on N^e for $r \geq r_*$.

Proof: According to Lemma 7 it is sufficient to show that the matrix $Df^T(x_e, y_e)Q + QDf(x_e, y_e) - E \cdot \text{Id}$ is positive definite for each $(x_e, y_e) \in N^e$. Put

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

We have

$$Df(x_{\mathrm{e}}, y_{\mathrm{e}}) = M_1 + \left(\frac{1}{8r^2} - \frac{3}{8}(x_{\mathrm{e}} + y_{\mathrm{e}})^2 \right) M_2 + \frac{1}{2r} M_3.$$

Easy calculations give

$$M_1^T Q = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \quad M_2^T Q = \begin{bmatrix} a & 1 \\ a & 1 \end{bmatrix}, \quad M_3^T Q = \begin{bmatrix} -a & -1 \\ a & 1 \end{bmatrix}.$$

We obtain

$$Df^T(x_{\mathrm{e}}, y_{\mathrm{e}})Q + QDf(x_{\mathrm{e}}, y_{\mathrm{e}}) =$$
$$\begin{bmatrix} 2a & 0 \\ 0 & 2 \end{bmatrix} + \left(\frac{1}{8r^2} - \frac{3}{8}(x_{\mathrm{e}} + y_{\mathrm{e}})^2 \right) \begin{bmatrix} 2a & 1+a \\ a+1 & 2 \end{bmatrix} +$$
$$\frac{1}{2r} \begin{bmatrix} -2a & a-1 \\ a-1 & 2 \end{bmatrix}.$$

Let us denote

$$w = \frac{1}{8r^2} - \frac{3}{8}(x_{\mathrm{e}} + y_{\mathrm{e}})^2. \tag{30}$$

Then

$$Df^T(x_{\mathrm{e}}, y_{\mathrm{e}})Q + QDf(x_{\mathrm{e}}, y_{\mathrm{e}}) - E \cdot \mathrm{Id} =$$
$$\begin{bmatrix} 2a\left(1 + w - \frac{1}{2r}\right) - E & w(a+1) + \frac{(a-1)}{2r} \\ w(a+1) + \frac{(a-1)}{2r} & 2\left(1 + w + \frac{1}{2r}\right) - E \end{bmatrix}.$$

It is immediate that for the positive definiteness of this matrix we need

$$1 + w - \frac{1}{2r} - E > 0.$$

From now on we set $a = 1$. The Geršgorin Theorem gives us the following sufficient condition for the positive definiteness of the matrix $Df^T(x_{\mathrm{e}}, y_{\mathrm{e}})Q + QDf(x_{\mathrm{e}}, y_{\mathrm{e}}) - E \cdot \mathrm{Id}$

$$2\left(1 + w - \frac{1}{2r}\right) - E > 2|w|,$$
$$2\left(1 + w + \frac{1}{2r}\right) - E > 2|w|.$$

We end up with the following condition

$$1 + w - \frac{1}{2r} > |w| + E/2.$$

Substituting back for w defined by (30) we obtain two inequalities that must be simultaneously satisfied

$$1 + \frac{1}{8r^2} - \frac{3}{8}(x_{\mathrm{e}} + y_{\mathrm{e}})^2 - \frac{1}{2r} > \frac{1}{8r^2} + \frac{3}{8}(x_{\mathrm{e}} + y_{\mathrm{e}})^2 + E/2,$$
$$1 - \frac{1}{2r} > \frac{3}{4}(x_{\mathrm{e}} + y_{\mathrm{e}})^2 + E/2.$$

19

Since for $(x_e, y_e) \in N^e$ there holds $|x_e + y_e| \leq (d_1^e + d_2^e)$ we obtain the following condition

$$1 > \frac{1}{2r} + \frac{3}{4}(d_1^e + d_2^e)^2 + E/2$$

and Lemma 7 implies the assertion (29). ∎

4.3   Estimate on constant m

Lemma 9 *Let $N^e = [-d_1^e, d_1^e] \times [-d_2^e, d_2^e]$ and put $d = d_1^e + d_2^e$. Then for $z = (x_e, y_e) \in N^e$ and $r \geq r_*$ there holds*

$$\left\| \left(\frac{\partial f}{\partial r}(r, z)\right)^T Q + Q \frac{\partial f}{\partial r}(r, z) \right\| \leq \frac{d}{2r_*^2}\left(1 + \frac{3}{4r_*}\right)\sqrt{a^2 + 1}.$$

Proof: Since we assumed that Q is diagonal it is easy to see that

$$\left\| \left(\frac{\partial f}{\partial r}(r, z)\right)^T Q + Q \frac{\partial f}{\partial r}(r, z) \right\| = 2\left\| Q \frac{\partial f}{\partial r}(r, z) \right\|.$$

We have

$$\frac{\partial f}{\partial r}(r, z) = \left(\frac{x_e - y_e}{2r^2} - \frac{3}{8r^3}(x_e + y_e)\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

hence

$$Q\frac{\partial f}{\partial r}(r, z) = \left(\frac{x_e - y_e}{2r^2} - \frac{3}{8r^3}(x_e + y_e)\right)\begin{bmatrix} a \\ 1 \end{bmatrix}.$$

Finally we obtain

$$\left\| Q\frac{\partial f}{\partial r}(r, z) \right\| \leq \left(\frac{d}{2r_*^2} + \frac{3d}{8r_*^3}\right)\sqrt{a^2 + 1} = \frac{d}{2r_*^2}\left(1 + \frac{3}{4r_*}\right)\sqrt{a^2 + 1}.$$

∎

4.4   Some numbers

Consider equation (26) and quadratic form (28) with $a = 1$ (this is the value used later in the actual proof). Lemma 6 and Lemma 8 guarantee that the set $N^e = [-d_1^e, d_1^e] \times [-d_2^e, d_2^e]$ is an isolating block for (26) satisfying the cone condition for all $r \geq r_*$ provided the following inequalities hold true:

$$\min\{d_1^e, d_2^e\} \;>\; d\left(\frac{1}{2r_*} + \frac{1}{8r_*^2} + \frac{d^2}{8}\right),$$

$$\frac{E}{2} + \frac{1}{2r_*} + \frac{3}{4}d^2 \;<\; 1,$$

where $d = d_1^e + d_2^e$.

For example, if $r_* = 2$ and $d_1^{\mathrm{e}} = d_2^{\mathrm{e}} = 1/4$ then the first inequality holds true. From Lemma 9 we get the bound $m \leq \frac{11}{128}\sqrt{2}$. Now the constant E should satisfy $E < 2 - 3/2d^2 - 1/r_* = 9/8$. Take for instance $E = 1$. This gives us a bound for the derivative $|x_s'(r)| \leq \frac{m}{E} = \frac{11\sqrt{2}}{128} \approx 0.121534$.

For r large the second condition gives an upper bound for $d < \frac{2}{3}\sqrt{3}$.

5 Estimates for the unstable manifold for $\rho \to -\infty$ for equation (1)

Let us rewrite here system (3)

$$w" - w/4 = e^{2\rho}(w - w^3). \tag{31}$$

We want to study the unstable manifold of $(0,0)$ for $\rho < 0$.

Passing formally to the limit $\rho \to -\infty$ in (31) we obtain the following linear system

$$w" - w/4 = 0. \tag{32}$$

Let $p = w'$ and let us introduce new coordinates $x_{\mathrm{b}}, y_{\mathrm{b}}$

$$\begin{cases} w = x_{\mathrm{b}} + y_{\mathrm{b}}, \\ p = \dfrac{x_{\mathrm{b}} - y_{\mathrm{b}}}{2}. \end{cases} \tag{33}$$

The inverse is given by

$$\begin{cases} x_{\mathrm{b}} = \dfrac{w + 2p}{2}, \\ y_{\mathrm{b}} = \dfrac{w - 2p}{2}. \end{cases}$$

In the coordinates $(x_{\mathrm{b}}, y_{\mathrm{b}})$ (31) becomes

$$\begin{cases} x_{\mathrm{b}}' = \dfrac{1}{2}x_{\mathrm{b}} + e^{2\rho}g(x_{\mathrm{b}}, y_{\mathrm{b}}), \\ y_{\mathrm{b}}' = -\dfrac{1}{2}y_{\mathrm{b}} - e^{2\rho}g(x_{\mathrm{b}}, y_{\mathrm{b}}), \end{cases} \tag{34}$$

where

$$g(x_{\mathrm{b}}, y_{\mathrm{b}}) = (x_{\mathrm{b}} + y_{\mathrm{b}}) - (x_{\mathrm{b}} + y_{\mathrm{b}})^3.$$

5.1 Isolating block

Lemma 10 *Let* $N^{\mathrm{b}} = [-d, d]^2$. *If* $d < 1/2$ *then* N^{b} *is an isolating block for (34) for any* ρ.

Proof: Observe that if $|x_{\mathrm{b}} + y_{\mathrm{b}}| \leq 1$ then $g(x_{\mathrm{b}}, y_{\mathrm{b}})$ has the same sign as $x_{\mathrm{b}} + y_{\mathrm{b}}$. The sign of $x_{\mathrm{b}} + y_{\mathrm{b}}$ coincides with the sign of the dominant coordinate. Therefore

$$x_{\mathrm{b}}g(x_{\mathrm{b}}, y_{\mathrm{b}}) > 0 \quad \text{for} \quad |x_{\mathrm{b}}| = d,$$
$$y_{\mathrm{b}}g(x_{\mathrm{b}}, y_{\mathrm{b}}) > 0 \quad \text{for} \quad |y_{\mathrm{b}}| = d.$$

This implies that

$$x'_{\mathrm{b}} x_{\mathrm{b}} = \frac{1}{2} x_{\mathrm{b}}^2/2 + e^{2\rho} x_{\mathrm{b}} g(x_{\mathrm{b}}, y_{\mathrm{b}}) \geq \frac{1}{2} x_{\mathrm{b}}^2/2 \quad \text{if} \quad |x_{\mathrm{b}}| = d,$$

$$-y'_{\mathrm{b}} y_{\mathrm{b}} = \frac{1}{2} y_{\mathrm{b}}^2/2 + e^{2\rho} y_{\mathrm{b}} g(x_{\mathrm{b}}, y_{\mathrm{b}}) \geq \frac{1}{2} y_{\mathrm{b}}^2/2 \quad \text{if} \quad |y_{\mathrm{b}}| = d.$$

∎

5.2 Cone condition

In this section we will derive inequalities that guarantee the cone condition on an isolating block for (34). Let us denote by f the vector field (34). We have

$$Df(x_{\mathrm{b}}, y_{\mathrm{b}}) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} + e^{2\rho}(1 - 3(x_{\mathrm{b}} + y_{\mathrm{b}})^2) \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

Put

$$Q = \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix}. \tag{35}$$

Lemma 11 *Assume that the isolating block for (34) has the form $N^{\mathrm{b}} = [-d_1^{\mathrm{b}}, d_1^{\mathrm{b}}] \times [-d_2^{\mathrm{b}}, d_2^{\mathrm{b}}]$. Put*

$$w = e^{2\rho}(1 - 3(x_{\mathrm{b}} + y_{\mathrm{b}})^2).$$

If

$$1 + 2w > E > 0, \tag{36}$$
$$a(1 + 2w)^2 - E(a+1)(1+2w) + E^2 > w^2(a+1)^2 \tag{37}$$

for $(x_{\mathrm{b}}, y_{\mathrm{b}}) \in N^{\mathrm{b}}$ and $\rho \leq \rho_0$ then the cone condition is satisfied on N^{b} for all $\rho \leq \rho_0$ for (34) with the constant E.

Proof: Denote by f the vector field (34). Direct computation gives

$$Df^T(x_{\mathrm{b}}, y_{\mathrm{b}})Q + QDf(x_{\mathrm{b}}, y_{\mathrm{b}}) =$$
$$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} + e^{2\rho}(1 - 3(x_{\mathrm{b}} + y_{\mathrm{b}})^2) \begin{bmatrix} 2a & a+1 \\ a+1 & 2 \end{bmatrix} =$$
$$\begin{bmatrix} a(1+2w) & w(a+1) \\ w(a+1) & 1+2w \end{bmatrix}.$$

We will apply the standard criterion for positive definiteness of matrices to the matrix $M = (Df^T(x_{\mathrm{b}}, y_{\mathrm{b}})Q + QDf(x_{\mathrm{b}}, y_{\mathrm{b}}) - E \cdot \mathrm{Id})$. We want $M_{22} > 0$ and $\det M > 0$. The first condition gives (36), the second reads

$$(a(1+2w) - E)(1 + 2w - E) - w^2(a+1)^2 > 0$$

which is equivalent to (37). ∎

Using the Geršgorin theorem [G, V] we can derive conditions, which might more suitable for easy estimates to be done by hand. The positive definiteness of

$$\begin{bmatrix} a(1+2w) & w(a+1) \\ w(a+1) & 1+2w \end{bmatrix}$$

is implied by the following two inequalities

$$a(1+2w) - w(a+1) > E, \tag{38}$$
$$1 + 2w - w(a+1) > E.$$

This leads to the following lemma.

Lemma 12 *Assume that $(d_1^b + d_2^b)^2 \leq \frac{1}{3}$, $0 < a \leq 1$ and the following condition is satisfied*

$$a(e^{2\rho_0} + 1) - e^{2\rho_0} > E.$$

Then the cone condition is satisfied on N^b for (34) for $\rho \leq \rho_0$ with the constant E.

Proof: Under the assumption on d_1^b and d_2^b we have

$$w = e^{2\rho}(1 - 3(x_b + y_b)^2) \geq 0.$$

Since $0 < a \leq 1$, the positive definiteness of $Df(x_b, y_b)^T Q + Q Df(x_b, y_b)$ is implied by only one condition (38) which reduces to

$$a(w+1) - w > E.$$

Observe that the range of w for $(x_b, y_b) \in N^b$ and $\rho \leq \rho_0$ is given by $0 < w \leq e^{2\rho_0} = w_0$. Since $a \leq 1$, the function $w \mapsto a(1+w) - w$ is non-increasing, hence it is enough to have this inequality for the largest possible value $w_0 = e^{2\rho_0}$. ∎

5.3   Estimation of constant m when $\rho \to -\infty$.

Lemma 13 *Denote by f the vector field (34). Assume that the isolating block for (34) has the form $N^b = [-d_1^b, d_1^b] \times [-d_2^b, d_2^b]$ with $d := \max\{d_1^b, d_2^b\} \leq 1/2$. Assume the quadratic form is diagonal as in (35). Then for $z = (x_b, y_b) \in N^b$ and $\rho \leq \rho_0$ there holds*

$$\left\| \left(\frac{\partial f}{\partial \rho}(\rho, z) \right)^T Q + Q \frac{\partial f}{\partial \rho}(\rho, z) \right\| \leq 8d e^{2\rho_0} \sqrt{a^2 + 1}.$$

Proof: For $z = (x_b, y_b) \in N^b$ we have

$$\frac{\partial f}{\partial \rho}(\rho, z) = 2e^{2\rho} g(x_b, y_b) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

23

Since Q is diagonal we get

$$Q\frac{\partial f}{\partial \rho}(\rho, z) = 2e^{2\rho}g(x_b, y_b)\begin{bmatrix} a \\ 1 \end{bmatrix}.$$

Given that

$$|g(x_b, y_b)| = |x_b + y_b|(1 - (x_b + y_b)^2) \leq |x_b + y_b| \leq 2d.$$

for $(x_b, y_b) \in N^b$ we obtain

$$\left\| Q\frac{\partial f}{\partial \rho}(\rho, z) \right\| \leq 4de^{2\rho_0}\sqrt{a^2 + 1}.$$

∎

5.4 Some numbers

We will show now that we can construct a quite large isolating block on which the cone condition is satisfied and still we can have good estimates on the derivative on parametrization of unstable set with respect to ρ.

From Lemma 10 the set $N^b = [-d, d]^2$ is an isolating block provided $d \leq 1/2$.

Lemma 12 guarantees the cone condition on N^b with constant E for $\rho \leq \rho_0$ provided the following two inequalities are satisfied

$$d \leq \frac{1}{2\sqrt{3}},$$
$$a(e^{2\rho_0} + 1) - e^{2\rho_0} > E.$$

In the case $a = 1$ (this is the value used in our computer assisted proof) the last inequality reduces to $E < 1$ and ρ_0 can be an arbitrary number. Finally, from Lemma 13 we obtain an estimate on the constant $m = 8de^{2\rho_0}\sqrt{2}$.

Explicit numbers satisfying all required inequalities:

- Put $d = 1/4$, $\rho_0 = \ln 2$, $a = 1$ and $E = 1/2$. Then we have $m = 8\sqrt{2}$, $|y_u'(\rho)| \leq m/E = 16\sqrt{2}$.

- Put $d = 1/4$, $\rho_0 = \ln 2^{-4}$, $a = 1$ and $E = 1/\sqrt{2}$. Then we have $m = \sqrt{2}/128$, $|y_u'(\rho)| \leq m/E = 1/64$.

6 Proof of Theorem 1

In this section we give a computer assisted proof of the existence of six geometrically different connecting orbits for equation (1), i.e. solutions satisfying (2). Each orbit makes different number of (half) revolutions around the separatrix of equation (24) before approaching the equilibrium point $(A, A') = (0, 0)$. Three of them converge to $(0, 0)$ from the right side $(A > 0)$ and three of them from the left side $(A < 0)$ - see Fig. 1 and Fig. 5.

Before we give the proof of Theorem 1 let us make several remarks about its content.
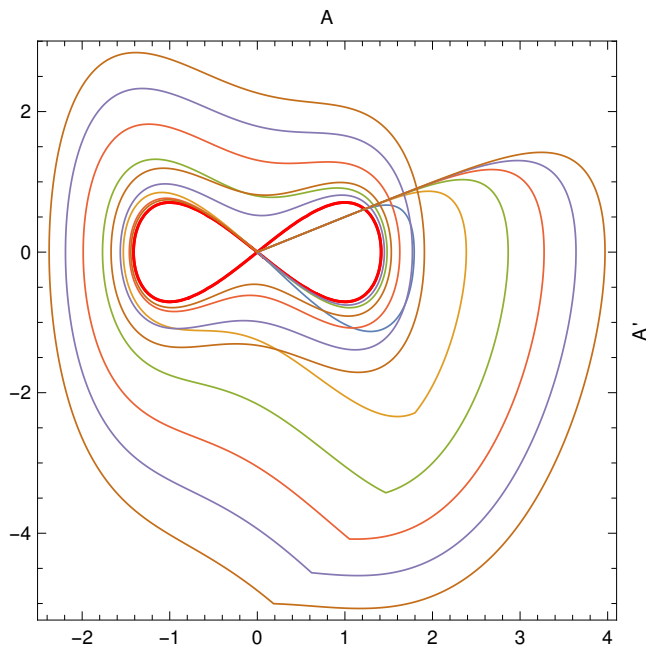
Figure 5: The six transverse connecting orbits resulting from Theorem 1. The trajectories are plotted in two time scales. For $r \geq 1$ the points $(A_n(r), A_n'(r))$ are shown while for $r \in (0, 1)$ we plot $(A_n(\rho), A_n'(\rho))$ with $\rho = \ln r$. This figure illustrates the convergence of orbits to $(0, 0)$ when $\rho \to -\infty$. The figure-eight curve in the middle is the separatrix for (24).

- W^u can be expressed using other independent variable $\rho = \ln r$ and $w(\rho) = A(e^\rho)$ (see Section 5) as follows.

$$W^u = \{(\rho_0, w_0, w_0')) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \mid$$
$$\text{such that the backward solution of (31) } w(\rho) \text{ with initial condition}$$
$$w(\rho_0) = w_0,\ w'(\rho_0) = w_0' \text{ satisfies } \lim_{\rho \to -\infty} (w(\rho), w'(\rho)) = 0\}.$$

In fact in the proof we use the above description of W^u. The expression for W^u given in the assertion of the theorem is obtained after we return to the original variables r and function $A(r)$.

- The statement about the number of local extremes means that the trajectory $r \to (A_n(r), A_n'(r))$ intersects transversally the axis $A' = 0$ exactly n times.

In Fig. 5 the six connecting orbits are shown in two time scales: r for $r \geq 1$ and $\rho = \ln r$ for $r \in (0, 1)$. For $r = 1$ they coincide, i.e.

$$(A(r = 1), A'(r = 1)) = (A(\rho = 0), A'(\rho = 0))$$

25

therefore they appear as continuous curves. This figure illustrates the convergence of orbits to $(0,0)$ when $\rho \to -\infty$.

The proof of Theorem 1 is based on the theorems introduced in the previous sections. Using a non-rigorous simulation (the bisection) we found good initial conditions for the connecting orbits. Then using validated numerics we verified the existence of isolating blocks, cone conditions and conditions for the intersection of the stable and unstable manifolds. The details will be given in the next sections.

In Sections 4 and 5 we have found analytically the isolating blocks on which the cone conditions are satisfied for the begin and end our connection. However, using validated numerics we can find much tighter bounds for the unstable/stable manifolds and similarly better bounds for the time-derivative of the parametrization of unstable/stable sets.

The input arguments to the algorithms that verify the existence of connecting orbit are the following

- n - number of intersections of the trajectory with the line $A' = 0$,

- $\hat{r}_n$ - an initial time,

- Δr_n - time range around $\hat{r}_n$.

These parameters are listed in (40) and found in nonrigorous simulation (bisection).

6.1   Isolating block and cone condition for $\rho \to -\infty$.

By $(x_{\mathrm{b}}, y_{\mathrm{b}})$ we denote coordinates that linearize the system at $\rho \to -\infty$ as defined in (33). We define an isolating block

$$N^{\mathrm{b}} = [-d_1^{\mathrm{b}}, d_1^{\mathrm{b}}] \times [-d_2^{\mathrm{b}}, d_2^{\mathrm{b}}] = [-0.125, 0.125] \times [-2.8 \cdot 10^{-8}, 2.8 \cdot 10^{-8}]. \quad (39)$$

Let us define the following constants

n	$\hat{r}_n$	Δr_n
1	0.003288250	$4 \cdot 10^{-7}$
2	0.001184020	$6 \cdot 10^{-8}$
3	0.000650050	$3 \cdot 10^{-8}$
4	0.000424204	$2 \cdot 10^{-8}$
5	0.000304427	$1 \cdot 10^{-8}$
6	0.000232050	$8 \cdot 10^{-9}$

$$(40)$$

Lemma 14 *Put $\rho_* = \ln(\hat{r}_1 + \Delta r_1)$, where $\hat{r}_1$ and Δr_1 are defined in (40).*

- *For $\rho \leq \rho_*$ set N^{b} is an isolating block for the system (34).*

- *For $\rho \leq \rho_*$ the cone condition is satisfied on N^{b} with quadratic form $Q(x_{\mathrm{b}}, y_{\mathrm{b}}) = x_{\mathrm{b}}^2 - y_{\mathrm{b}}^2$ and thus for all $\rho \leq \rho_*$ the set*

$$W_{\rho, N^{\mathrm{b}}}^u(0,0) = \{(x_{\mathrm{b}}, y_u(\rho, x_{\mathrm{b}})) \,|\, x_{\mathrm{b}} \in [-d_1^{\mathrm{b}}, d_1^{\mathrm{b}}]\}$$

is a horizontal disc satisfying the cone condition.

- *The following estimate holds:*

$$\left| \frac{\partial y_u}{\partial \rho}(\rho, x_{\mathrm{b}}) \right| \leq 7.65 \cdot 10^{-6}.$$

Proof: Let $f(\rho, x_{\mathrm{b}}, y_{\mathrm{b}}) = (f_1(\rho, x_{\mathrm{b}}, y_{\mathrm{b}}), f_2(\rho, x_{\mathrm{b}}, y_{\mathrm{b}}))$ be the vector field as defined in (34) and put $f_\rho = f(\rho, \cdot, \cdot)$. Let $d_1^{\mathrm{b}}, d_2^{\mathrm{b}}$ be the sizes of N^{b} – see (39). Direct evaluation in interval arithmetics gives the following inequalities

$$
\begin{aligned}
f_1((-\infty, \rho_*] \times \{d_1^{\mathrm{b}}\} \times [-d_2^{\mathrm{b}}, d_2^{\mathrm{b}}]) &\geq & 0.0625 > 0, \\
f_1((-\infty, \rho_*] \times \{-d_1^{\mathrm{b}}\} \times [-d_2^{\mathrm{b}}, d_2^{\mathrm{b}}]) &\leq & -0.0625 < 0, \\
f_2((-\infty, \rho_*] \times [-d_1^{\mathrm{b}}, d_1^{\mathrm{b}}] \times \{d_2^{\mathrm{b}}\}) &\leq & -4.8127929800192025 \cdot 10^{-8} < 0, \\
f_2((-\infty, \rho_*] \times [-d_1^{\mathrm{b}}, d_1^{\mathrm{b}}] \times \{-d_2^{\mathrm{b}}\}) &\geq & 4.8127929800192025 \cdot 10^{-8} > 0.
\end{aligned}
$$

This proves that N^{b} is an isolating block for all $\rho \leq \rho_*$.

Let us fix $\rho \in (-\infty, \rho_*]$ and recall that the quadratic form we are using is $Q(x_{\mathrm{b}}, y_{\mathrm{b}}) = x_{\mathrm{b}}^2 - y_{\mathrm{b}}^2$. Since $d_1^{\mathrm{b}} + d_2^{\mathrm{b}} \leq \frac{\sqrt{3}}{3}$ from Lemma 12 we get that the cone condition is satisfied with any constant $E < 1$.

Finally, from Lemma 13 and Lemma 5 and passing to the limit with $E \to 1^-$ we get that

$$
\begin{aligned}
\left| \frac{\partial y_u}{\partial \rho}(\rho, x_{\mathrm{b}}) \right| &\leq \frac{\left\| \frac{\partial f}{\partial \rho}((-\infty, \rho_*] \times N^{\mathrm{b}})^T Q + Q \frac{\partial f}{\partial \rho}((-\infty, \rho_*] \times N^{\mathrm{b}}) \right\|}{E} \\
&\leq \frac{4\sqrt{2} e^{2\rho_*} |d_1^{\mathrm{b}} + d_2^{\mathrm{b}}|}{E} \quad \overset{E \to 1^-}{\longrightarrow} \quad 4\sqrt{2} e^{2\rho_*} |d_1^{\mathrm{b}} + d_2^{\mathrm{b}}| \\
&= 540761062255450812 \cdot 10^{-23} \sqrt{2} < 7.65 \cdot 10^{-6}.
\end{aligned}
$$

∎

6.2 Isolating block and cone condition for $r \to \infty$.

Let us consider an isolating block for $r \to \infty$. Put

$$N^{\mathrm{e}} = [-d_1^{\mathrm{e}}, d_1^{\mathrm{e}}] \times [-d_2^{\mathrm{e}}, d_2^{\mathrm{e}}] = [-0.0015, 0.0015] \times [-0.01, 0.01]. \qquad (41)$$

Lemma 15 *Let $r_* = 6$.*

- *For $r \geq r_*$ the set N^{e} is an isolating block for the system (24).*

- *For $r \geq r_*$ the cone condition is satisfied on N^{e} with quadratic form $Q(x_{\mathrm{e}}, y_{\mathrm{e}}) = x_{\mathrm{e}}^2 - y_{\mathrm{e}}^2$ and thus for all $r \geq r_*$ the set*

$$W_{r, N^{\mathrm{e}}}^s(0,0) = \{(x_s(r, y_{\mathrm{e}}), y_{\mathrm{e}}) \,|\, y_{\mathrm{e}} \in [-d_2^{\mathrm{e}}, d_2^{\mathrm{e}}]\}$$

  is a vertical disc satisfying the cone condition.

• *The following estimate holds:*

$$\left|\frac{\partial x_s}{\partial r}(\rho, y_{\mathrm{e}})\right| \leq 0.000252.$$

Proof: Let $f(r, x_{\mathrm{e}}, y_{\mathrm{e}}) = (f_1(r, x_{\mathrm{e}}, y_{\mathrm{e}}), f_2(r, x_{\mathrm{e}}, y_{\mathrm{e}}))$ be the vector field as defined in (24) and let $d_1^{\mathrm{e}}, d_2^{\mathrm{e}}$ be the sizes of N^{e} – see (41). Direct evaluation in interval arithmetics gives the following inequalities

$$
\begin{aligned}
f_1([r_*, \infty) \times \{d_1^{\mathrm{e}}\} \times [-d_2^{\mathrm{e}}, d_2^{\mathrm{e}}]) &\geq & 0.00051215277777777761 > 0, \\
f_1([r_*, \infty) \times \{-d_1^{\mathrm{e}}\} \times [-d_2^{\mathrm{e}}, d_2^{\mathrm{e}}]) &\leq & -0.00051215277777777761 < 0, \\
f_2([r_*, \infty) \times [-d_1^{\mathrm{e}}, d_1^{\mathrm{e}}] \times \{d_2^{\mathrm{e}}\}) &\leq & -0.010737706706597221 < 0, \\
f_2([r_*, \infty) \times [-d_1^{\mathrm{e}}, d_1^{\mathrm{e}}] \times \{-d_2^{\mathrm{e}}\}) &\geq & 0.010737706706597221 > 0.
\end{aligned}
$$

This proves that N^{e} is an isolating block for all $r \geq r_*$.

From the proof of Lemma 8 it follows that the cone condition is satisfied on the set N^{e} for $r \geq r_*$ and with $a = 1$, $E > 0$ provided the following inequality holds true

$$E < 2 - \frac{1}{r_*} - \frac{3}{2}(d_1^{\mathrm{e}} + d_2^{\mathrm{e}})^2$$

Substituting $r_* = 6$ and $d_1^{\mathrm{e}}, d_2^{\mathrm{e}}$ form (41) we get the upper bound

$$E < E_0 = \frac{43995239}{24000000} \approx 1.8331349583333332.$$

Finally, from Lemma 9 and Lemma 5 we get that

$$\left|\frac{\partial x_s}{\partial r}(r, y_{\mathrm{e}})\right| \leq \frac{\left\|\frac{\partial f}{\partial r}([r_*, \infty) \times N^{\mathrm{e}})^T Q + Q \frac{\partial f}{\partial r}([r_*, \infty) \times N^{\mathrm{e}})\right\|}{E}$$

$$\leq \frac{\sqrt{2}(d_1^{\mathrm{e}} + d_2^{\mathrm{e}})}{r_*^2}\left(1 + \frac{3}{4r_*^2}\right) / E.$$

Passing to the limit with $E \to E_0^-$ we obtain

$$\left|\frac{\partial x_s}{\partial r}(r, y_{\mathrm{e}})\right| \leq \frac{\sqrt{2}(d_1^{\mathrm{e}} + d_2^{\mathrm{e}})}{r_*^2}\left(1 + \frac{3}{4r_*^2}\right) / E_0 = \frac{140875}{791914302}\sqrt{2} < 0.000252.$$

∎

6.3 Shooting between manifolds.

Let us define three Poincaré sections. The first section is expressed in coordinates $(r, x_{\mathrm{b}}, y_{\mathrm{b}})$ – see (33) – that linearize the system at $\rho \to -\infty$. The section contains one of the exit edges of the set N^{b}, namely

$$\Pi^{\mathrm{b}} = \left\{(r, x_{\mathrm{b}}, y_{\mathrm{b}}) \,|\, x_{\mathrm{b}} = d_1^{\mathrm{b}}\right\}.$$

We will use (r, y_b) coordinates to describe points on Π^b.

The two remaining sections are expressed in coordinates that define the set N^e and each of them contains one of the entrance edges of this set

$$\Pi^e_\pm = \{(r, x_e, y_e) \mid y_e = \pm d^e_2\}.$$

We will use (r, x_e) coordinates to describe points on $\Pi^e_\pm$ when the sign will be clear from the context. By $P_- : \Pi^b \to \Pi^e_-$ and $P_+ : \Pi^b \to \Pi^e_+$ we denote two Poincaré maps. By $P^n_\pm(r, y_b)$ we denote the n-th itersection of the trajectory of (1) starting at (r, d^b_1, y_b) with the section $\Pi^e_\pm$. In what follows we will always use odd number of intersections for the mapping P^n_- and even for the mapping P^n_+. Therefore we will skip the subscript $\pm$ to simplify the notation and write just P^n.

Lemma 16 *Let $\hat{r}_n$, Δr_n, $n = 1, \ldots, 6$ be as defined in (40) and put*

$$\begin{aligned}
r^-_n &= \hat{r}_n - \Delta r_n \\
r^+_n &= \hat{r}_n + \Delta r_n \\
D^b_2 &= [-d^b_2, d^b_2].
\end{aligned}$$

Then for $n = 1, \ldots, 6$ the mapping P^{n+1} is well defined and smooth on the set

$$[r^-_n, r^+_n] \times [-d^b_2, d^b_2].$$

Moreover

$$\begin{aligned}
P^{n+1}(\{r^\pm_n\} \times D^b_2) &\subset \{(r, x_e) : |x_e| > d^e_1, r \geq r_* = 6\}, \\
\pi_x P^{n+1}(r^-_n, y_b) \cdot \pi_x P^{n+1}(r^+_n, y_b) &< 0 \quad \text{for } y_b \in D^b_2
\end{aligned}$$

and each trajectory of a point $(r_b, y_b) \in [r^-_n, r^+_n] \times [-d^b_2, d^b_2]$ intersects the axis $A' = 0$ exactly n times before reaching the section $\Pi^e_{(-1)^n}$.

Proof: The proof is computer assisted and uses an rigorous ODE solver from the CAPD library [CAPD]. In (42) we give rigorous bounds for the return time, i.e. for all $(r_b, y_b) \in [r^-_n, r^+_n] \times [-d^b_2, d^b_2]$ the image $(r_e, x_e) := P^{(n+1)}(r_b, y_b)$ exists and r_e belongs to the corresponding interval listed in (42). This shows that such an intersection occurs always for $r_e \geq r_* = 6$.

n	bound for the return time r_e	
1	$[6.5694270711914049, 6.8663028711914071]$	
2	$[9.547364685097655, 9.8754898050976578]$	
3	$[12.63188037430908, 12.975630434309084]$	(42)
4	$[15.467339314615232, 15.811089354615238]$	
5	$[18.659449295387446, 19.009330830371958]$	
6	$[21.645791630860828, 21.989541646860836]$	

In (43) we list estimates which establish the remaining inequalities. We see that the x-coordinate of $P^{n+1}(\{r^\pm_n\} \times D^b_2)$ has opposite sign for r^-_n and r^+_n and in
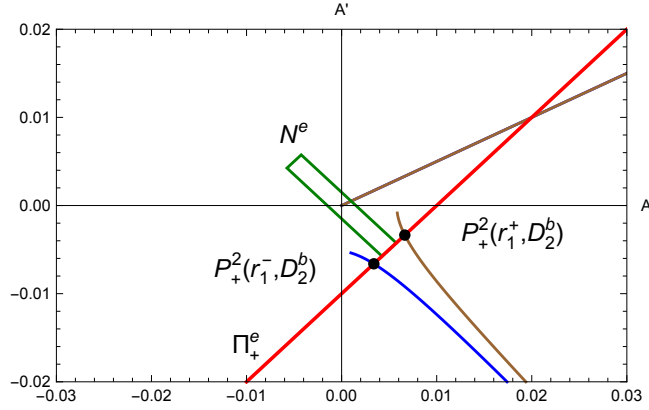
29

Figure 6: The set N^e, Poincaré section Π^e_+ and an indication that $P^2_+(r^-_1, D^b_2)$ and $P^2_+(r^+_1, D^b_2)$ are mapped to opposite sides of the set N^e.

each case its absolute value is bigger than $d^e_1 = 0.0015$.

$$
\begin{array}{c|c|c}
n & \pi_x P^{n+1}(\{r^-_n\} \times D^b_2) & \pi_x P^{n+1}(\{r^+_n\} \times D^b_2) \\
\hline
1 & -0.0032[19, 26] & 0.0032[89, 97] \\
2 & 0.00189[2, 8] & -0.0018[86, 93] \\
3 & -0.0019[87, 92] & 0.0019[37, 43] \\
4 & 0.00209[5, 9] & -0.00221[0, 5] \\
5 & -0.00155[5, 9] & 0.0015[78, 82] \\
6 & 0.00173[1, 4] & -0.0016[57, 61]
\end{array}
\tag{43}
$$

In Fig. 6 we show the isolating block N^e, Poincaré section Π^e_+ and by black dots we marked the image $P^2_+(\{r^\pm_1\} \times D^b_2)$. ∎

Proof of Theorem 1:

Recall that we used (r_b, y_b) coordinates on section Π^b and (r_e, x_e) coordinates on Π^e_+. From Lemmas 14 and 15 we know that

$$
W^u_{\rho, N^b}(0, 0) = \{(x_b, y_u(\rho, x_b)) \,|\, x_b \in [-d^b_1, d^b_1]\},
$$
$$
W^s_{r, N^e}(0, 0) = \{(x_s(r, y_e), y_e) \,|\, y_e \in [-d^e_2, d^e_2]\},
$$

where $y_u(\cdot, \cdot)$, $x_s(\cdot, \cdot)$ are smooth functions. Put

$$
\tilde{y}_u(r_b, x_b) := y_u(\ln r_b, x_b)
$$

and consider the mapping

$$
F_n(r_b) = \pi_{x_e} P^{n+1}\left(r_b, \tilde{y}_u(r_b, d^b_1)\right) - x_s\left(\pi_r P^{n+1}(r_b, \tilde{y}_u(r_b, d^b_1)), d^e_2\right).
$$

Zeros of this function correspond to connecting orbits we are searching for. We will show that for $n = 1, \ldots, 6$ this function has a unique zero in $[r^-_n, r^+_n]$.

From Lemma 16 the function F_n is continuous and $F_n(r_n^-)F_n(r_n^+) < 0$, because $|x_s\left(\pi_r P^{n+1}(r_\mathrm{b}, \tilde{y}_u(r_\mathrm{b}, d_1^\mathrm{b})), d_2^\mathrm{e}\right)| \le d_1^\mathrm{e}$. This proves that F_n has zero in $[r_n^-, r_n^+]$. For uniqueness it is enough to show that $F_n'(r) \ne 0$ for $r \in [r_n^-, r_n^+]$.

The proof is computer assisted and we will give details for $n = 1$, only. For $n = 2, \ldots, 6$ we will give the necessary estimates. Using the CAPD library [CAPD] we computed a rigorous bound for the derivative of the return map P_+^2 on the set $W = [r_1^-, r_1^+] \times [-d_2^\mathrm{b}, d_2^\mathrm{b}]$ and we got

$$
\begin{aligned}
DP_+^2(W) &= \begin{bmatrix} \frac{\partial \pi_r P_+^2}{\partial r_\mathrm{b}}(W) & \frac{\partial \pi_r P_+^2}{\partial y_\mathrm{b}}(W) \\ \frac{\partial \pi_{x_e} P_+^2}{\partial r_\mathrm{b}}(W) & \frac{\partial \pi_{x_e} P_+^2}{\partial y_\mathrm{b}}(W) \end{bmatrix} \\
&\subset \begin{bmatrix} [-292366, 565586] & [-58.3544, 88.8462] \\ [1538.59, 16826.7] & [-0.253485, 2.36953] \end{bmatrix}.
\end{aligned}
$$

From Lemma 14 and Lemma 15 we have bounds for the partial derivatives of the parametrization of invariant sets with respect to time variable, namely

$$
\left| \frac{\partial y_u}{\partial \rho}(\rho, d_1^\mathrm{b}) \right| \le 7.65 \cdot 10^{-6},
$$

$$
\left| \frac{\partial x_s}{\partial r_\mathrm{e}}(r_\mathrm{e}, d_2^\mathrm{e}) \right| \le 0.000252.
$$

Taking into account time rescaling $\rho = \ln r_\mathrm{b}$ we obtain the estimate

$$
\left| \frac{\partial}{\partial r_\mathrm{b}} \tilde{y}_u(r_\mathrm{b}, d_1^\mathrm{b}) \right| = \left| \frac{\partial}{\partial r_\mathrm{b}} y_u(\ln r_\mathrm{b}, d_1^\mathrm{b}) \right| \le \frac{7.65 \cdot 10^{-6}}{r_1^-} = \frac{51}{21919} < 0.00233.
$$

Put $[x_s'] := 0.000252 \cdot [-1, 1]$ and $[\tilde{y}_u'] := 7.65 \cdot 10^{-6} \cdot [-1, 1]$.

Gathering this together we obtain a rigorous bound for the derivative

$$
\begin{aligned}
F_1'([r_1^-, r_1^+]) &\subset \frac{\partial \pi_{x_e} P_+^2}{\partial r_\mathrm{b}}(W) + \frac{\partial \pi_{x_e} P_+^2}{\partial y_\mathrm{b}}(W)[\tilde{y}_u'] \\
&\quad - [x_s']\left(\frac{\partial \pi_r P_+^2}{\partial r_\mathrm{b}}(W) + \frac{\partial \pi_r P_+^2}{\partial y_\mathrm{b}}(W)[\tilde{y}_u'] \right) \\
&\subset [1538.59, 16826.7] + [-0.253485, 2.36953][\tilde{y}_u'] \\
&\quad - [x_s']([-292366, 565586] + [-58.3544, 88.8462][\tilde{y}_u']) \\
&\subset [1396, 16970],
\end{aligned} \tag{44}
$$

which is nonzero (see Remark 17 for short comments about the possibility for obtaining more precise bounds).

In a similar way we computed this derivative for $n = 2, \ldots, 6$ and we got the following bounds

$$
\begin{aligned}
F_2'([r_2^-, r_2^+]) &\subset [-56203.7, -12850.2], \\
F_3'([r_3^-, r_3^+]) &\subset [16342.1, 137481], \\
F_4'([r_4^-, r_4^+]) &\subset [-183339, -24681.3], \\
F_5'([r_5^-, r_5^+]) &\subset [39626.1, 307922], \\
F_6'([r_6^-, r_6^+]) &\subset [-435119, -1138.93].
\end{aligned} \tag{45}
$$

It remains to show the transversality of A_n.

Observe that till now we established the following facts

- Let $W^u_{(\rho \leq \rho_*), N^b} \subset (-\infty, \rho_*] \times N^b$ be a set consisting of all points (ρ_0, z_0), such that there exists a solution $z(\rho)$ for $\rho \leq \rho_0$ of equation (31) (in coordinates in which N^b is defined this is the system (34)), such that $z(\rho_0) = z_0$, $z((-\infty, \rho_0] \subset N^b$ and $\lim_{\rho \to -\infty} z(\rho) = (0, 0)$.

  We have proved that there exists a smooth function $y_u(\rho, x_b)$, such that

  $$W^u_{(\rho \leq \rho_*), N^b} = \{(\rho, x_b, y_u(\rho, x_b)), \mid \rho \in (-\infty, \rho_*], x_b \in [-d_1^b, d_1^b]\}$$

- Let $W^s_{(r \geq r_*), N^e} \subset [r_*, \infty) \times N^e$ be a set consisting of all points (r_0, z_0), such that there exists a solution $z(r)$ for $r \geq r_0$ of equation (24) (in coordinates in which N^e is defined this is the system (26)), such that $z(r_0) = z_0$, $z[r_0, \infty) \subset N_e$ and $\lim_{r \to \infty} z(r) = (0, 0)$.

  We have proved that there exists a smooth function $x_s(r, y_e)$, such that

  $$W^s_{(r \geq r_*), N^e} = \{(r, x_s(r, y_e), y_e), \mid r \in [r_*, \infty), y_e \in [-d_2^e, d_2^e]\}.$$

The sets $W^u_{(\rho \leq \rho_*), N^b}$ and $W^s_{(r \geq r_*), N^e}$ are two dimensional surfaces in the extended phase space $\mathbb{R} \times \mathbb{R}^2$ (or $[0, \infty) \times \mathbb{R}^2$ depending whether we use the variable ρ or r). These surfaces can be 'globalized' to invariant sets W^u and W^s, by applying to $W^u_{(\rho \leq \rho_*), N^b}$ and $W^s_{(r \geq r_*), N^e}$ the flow induced by (1). In this way we obtain two-dimensional immersed invariant manifolds.

Each of the solutions A_n whose existence we already established belongs to $W^u \cap W^s$. We want to prove that the intersection along such solution is transversal.

Let $\gamma(r) := (r, A_n(r), A_n'(r))$ for some $n = 1, \ldots, 6$ be one of our connecting orbits. Let $p_b = (\bar{r}_b, d_1^b, \tilde{y}_u(\bar{r}_b, d_1^b))$ and $p_e = (\bar{r}_e, x_s(\bar{r}_e, d_2^e), d_2^e)$ be the points on γ on sections Π^b (i.e. $x_b = d_1^b$) and Π^e (i.e. $y_e = d_2^e$), respectively. To prove the transversality it is enough to show that the tangent spaces to W^u and W^s at p_e satisfy
$$T_{p_e} W^u + T_{p_e} W^s = \mathbb{R}^3.$$

Since the point p_e is in the isolating block N^e in coordinates (x_e, y_e) we have

$$T_{p_e} W^s = T_{p_e} W^s_{(r \geq r_*), N^e} = \text{span}\left\{\gamma'(\bar{r}_e), \left(1, \frac{\partial x_s}{\partial r}(\bar{r}_e, d_2^e), 0\right)\right\}$$

and these vectors are linearly independent because the last coordinate of $\gamma'(\bar{r}_e)$ is nonzero. In the above formula the vector $\gamma'(\bar{r}_e)$ is the direction of the vector field (and the curve γ). The second vector

$$g_1 = \left(1, \frac{\partial x_s}{\partial r}(\bar{r}_e, d_2^e), 0\right)$$

spans $T_{p_e}\left(W^s_{(r \geq r_*), N^e} \cap \Pi^e\right)$, which is one dimensional.

$T_{p_e} W^u$ is obtained from $T_{p_b} W^u_{(\rho \leq \rho^*), N^b}$, as follows. $T_{p_b} W^u_{(\rho \leq \rho^*), N^b}$ is spanned by $\gamma'(\bar{r}_b)$ and

$$g_2 = \left(1, 0, \frac{\partial \tilde{y}_u}{\partial r}(\bar{r}_b, d_1^b) \right),$$

which spans the one-dimensional space $T_{p_b} \left(W^u_{(\rho \leq \rho_*), N^b} \cap \Pi^b \right)$.

Put $\tau = \bar{r}_e - \bar{r}_b$ which is the transition time for P^{n+1} at p_e. The shift along the trajectory of γ by τ maps $\gamma'(\bar{r}_b)$ to $\gamma'(\bar{r}_e)$. The Poincaré map P^{n+1} acts on g_2 to produce a vector $\tilde{g}_2$ as follows

$$
\begin{aligned}
\pi_r \tilde{g}_2 &= \frac{\partial \pi_r P^{n+1}}{\partial r}(\bar{r}_b, \tilde{y}_u(\bar{r}_b, d_1^b)) + \frac{\partial \pi_r P^{n+1}}{\partial y_b}(\bar{r}_b, \tilde{y}_u(\bar{r}_b, d_1^b)) \frac{\partial \tilde{y}_u}{\partial r}(\bar{r}_b, d_1^b), \\
\pi_{x_e} \tilde{g}_2 &= \frac{\partial \pi_{x_e} P^{n+1}}{\partial r}(\bar{r}_b, \tilde{y}_u(\bar{r}_b, d_1^b)) + \frac{\partial \pi_{x_e} P^{n+1}}{\partial y_b}(\bar{r}_b, \tilde{y}_u(\bar{r}_b, d_1^b)) \frac{\partial \tilde{y}_u}{\partial r}(\bar{r}_b, d_1^b), \\
\pi_{y_e} \tilde{g}_2 &= 0.
\end{aligned}
$$

We have

$$T_{p_e} W^u = \operatorname{span} \left\{ \gamma'(\bar{r}_e), \tilde{g}_2 \right\}.$$

Therefore we need to show that the vectors $\gamma'(\bar{r}_e), g_1, \tilde{g}_2$ are linearly independent. Observe that $\pi_{y_e} \gamma'(\bar{r}_e) \neq 0$, because the section Π^e is transversal, and $\pi_{y_e} g_1 = \pi_{y_e} \tilde{g}_2 = 0$. Therefore, it is enough to check that the projections of $g_1, \tilde{g}_2$ on (r, x_e) subspace are linearly independent. This means the following determinant is nonzero (we dropped arguments in the partial derivatives)

$$D = \det \begin{bmatrix} 1, & \frac{\partial \pi_r P^{n+1}}{\partial r} + \frac{\partial \pi_r P^{n+1}}{\partial y} \frac{\partial \tilde{y}_u}{\partial r} \\ \frac{\partial x_e}{\partial r}, & \frac{\partial \pi_{x_e} P^{n+1}}{\partial r} + \frac{\partial \pi_{x_e} P^{n+1}}{\partial y} \frac{\partial \tilde{y}_u}{\partial r} \end{bmatrix}.$$

Observe that this leads to the following condition

$$D = \frac{\partial \pi_{x_e} P^{n+1}}{\partial r} + \frac{\partial \pi_{x_e} P^{n+1}}{\partial y} \frac{\partial \tilde{y}_u}{\partial r} - \frac{\partial x_e}{\partial r} \left(\frac{\partial \pi_r P^{n+1}}{\partial r} + \frac{\partial \pi_r P^{n+1}}{\partial y} \frac{\partial \tilde{y}_u}{\partial r} \right) \neq 0.$$

We have

$$
\begin{aligned}
\frac{\partial \pi_{x_e} P^{n+1}}{\partial r} + \frac{\partial \pi_{x_e} P^{n+1}}{\partial y} \frac{\partial \tilde{y}_u}{\partial r} &= \frac{d}{dr} \pi_{x_e} P^{n+1}(r, \tilde{y}_u(r, d_2^b)), \\
\frac{\partial \pi_r P^{n+1}}{\partial r} + \frac{\partial \pi_r P^{n+1}}{\partial y} \frac{\partial \tilde{y}_u}{\partial r} &= \frac{d}{dr} \pi_r P^{n+1}(r, \tilde{y}_u(r, d_2^b)).
\end{aligned}
$$

therefore

$$D = F_n'(\bar{r}_e) \neq 0$$

hence the intersection is transversal. ∎

Remark 17 *The estimation of F_1' given by (44) is very rough, but it is sufficient for our purposes. It is easy to see that the main reason for such big diameter of our estimate for F_1' comes from our bound for $\frac{\partial \pi_{x_e} P^2}{\partial r_b}$. This number can be non-rigorously estimated by taking finite differences using data from (43) (first row) and Δr_1 from (40) to obtain*

$$\frac{\pi_{x_e} P^2(r_1^+, D_2^b) - \pi_{x_e} P^2(x_1^-, D_2^b)}{2\Delta r_1} \approx (0.003294 + 0.003223)/(8 \cdot 10^{-7}) = 8146.25.$$

Subdividing Δr_1 onto 100 pieces we got a rigorous estimate on the derivative

$$F_1'([r_1^-, r_1^+]) \subset [7902.54, 8399.24].$$

From the above computations it is clear that the key ingredients in obtaining nonzero derivative $F'([r_1^-, r_1^+])$ are sharp bounds for $[x_e']$ obtained from the cone condition and the transversality coming from dynamics, i.e. large in absolute value coefficient $\frac{\partial \pi_{x_e} P_\pm^2}{\partial r_b}(W)$.

6.4 Implementation notes.

A short C++11 program that realizes part of a computer assisted proof of Theorem 1 is available from [W]. All the inequalities, i.e. return times (42), shooting (43) and estimation on derivatives (45), regarding all six connecting orbits are checked within 0.7 second on a laptop type computer with Intel Core i7 2GHz processor. The program has been compiled and tested under Ubuntu OS with gcc-4.8.1 and gcc-4.9.1, OSX 10 with clang 6.0 and MS Windows 7 with gcc-4.8.1.

This program does not contain nonrigorous routines for finding good candidates for connecting orbits. They have been found by a simple bisection algorithm. The sizes of sets (see (40)) were adjusted by hand so that no subdivisions are necessary when computing derivatives F_n'.

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