Rigorous verification of cocoon bifurcations in the Michelson system

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Dedicated to the sixtieth birthday of Professor Freddy Dumortier

Abstract

We prove the existence of *cocoon* bifurcations for the Michelson system

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = c^2 - y - \frac{1}{2}x^2,$$

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where $(x, y, z) \in \mathbb{R}^3$ and $c \in \mathbb{R}_+$ is a parameter, based on the theory given in [5]. The main difficulty lies in the verification of the (topological) transversality of some invariant manifolds in the system. The proof is computer assisted and combines topological tools including covering relations and the smooth ones using the cone conditions. These new techniques developed in this paper will have broader applicability to similar global bifurcation problems.

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1 Introduction

The goal of this paper is to rigorously prove the existence of *cocoon* bifurcation [5] for the Michelson system [12]

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= c^2 - y - \frac{1}{2}x^2 \end{aligned} \tag{1}$$

The Michelson system arises, on one hand, as the travelling wave equation of a PDE called the Kuramoto-Shivashinsky equation, and on the other hand, as a part of the limit family of the unfolding of a codimension three nilpotent vector field singularity, see [4, 5] and references therein for the details.

The Michelson system (1) has the following basic properties among others:

- The system is reversible with respect to the involution [10]

$$R(x, y, z) = (-x, y, -z). \tag{2}$$

  This means that the transformation of the form

$$(t, (x, y, z)) \mapsto (-t, R(x, y, z))$$

  preserves orbits of (1).

- For $c > 0$ there are only two equilibrium points at

$$x_\pm = (\pm\sqrt{2}c, 0, 0), \tag{3}$$

  both are of saddle-focus type with $\dim(W^u(x_+)) = \dim(W^s(x_-)) = 2$.

It was proven in [13] that, for c sufficiently large, there exists for (1) a unique transverse heteroclinic orbit connecting x_+ and x_-, given by the intersection of two-dimensional $W^u(x_+)$ and $W^s(x_-)$, and the equilibrium points together with the heteroclinic orbit form the maximal bounded invariant set for (1).

When the parameter c decreases, the numerical results in [12, 11] show that (1) exhibits an infinite sequence of heteroclinic bifurcations due to the tangency of $W^u(x_+)$ and $W^s(x_-)$, each of which creates a pair of new transverse heteroclinic orbits. Also numerically, the sequence of values of c for which these bifurcations occur converges to $\bar{c} \approx 1.2662$. At this value, there appears a saddle-node bifurcation that creates a periodic orbit γ_*. The sequence of bifurcations that appear before and after $c = \bar{c}$ was studied by Lau[11] mainly numerically, and was called the "cocoon" bifurcation, because of the shape of the invariant manifolds controlling the process.

Dumortier, Ibañez and the first author of this paper [5] studied the cocoon bifurcation from a theoretical and more general point of view, and explained its occurrence as a consequence of the presence of an organizing center called the *cusp-transverse heteroclinic chain*, which is defined as follows:

Let X_λ be a one-parameter family of vector fields on $\mathbb{R}^3$ having the following properties:

(H1) each of the vector fields X_λ is reversible with respect to the linear involution R with $\dim(\mathrm{Fix}(R)) = 1$, where $\mathrm{Fix}(R)$ is the fixed point subspace of R

(H2) There exist two hyperbolic equilibrium points $x_\pm \notin \mathrm{Fix}(R)$ which are symmetric under the involution R and such that
$\dim(W^u(x_+)) = \dim(W^s(x_-)) = 2$.

Definition 1 [5, Definition 1.3] Under the conditions (H1) and (H2), we say the family X_λ exhibits a *cocooning cascade of heteroclinic tangencies* centered at λ_*, if there is a closed solid torus T with $x_\pm \notin T$ and a monotone infinite sequence of parameters λ_n converging to λ_*, for which the corresponding vector field X_{λ_n} has a tangency of $W^u(x_+)$ and $W^s(x_-)$ such that the heteroclinic tangency orbit intersects with T and has its length within T tending to infinity as $n \to \infty$.

Definition 2 [5, Definition 1.4] A family of vector fields X_λ on $\mathbb{R}^3$ satisfying (H1) and (H2) is said to have a *cusp-transverse heteroclinic chain* at $\lambda = \lambda_0$, if the following three conditions hold:

(C1) X_{λ_0} has a saddle-node periodic orbit γ_* which is symmetric under the involution R.

Here the saddle-node periodic orbit is meant by a periodic orbit whose Poincaré map has the unity as its eigenvalue. Under the presence of the reversibility, this implies that the other eigenvalue of the linearized Poincaré map is also the unity. See the discussion in the beginning of Section 7.

(C2) The saddle-node periodic orbit γ_* is generic and generically unfolded in X_λ under the reversibility with respect to R. Here the genericity means that some of the derivatives of the Poincaré map for the saddle-node periodic orbit are non-zero, see Section 7 and [5] for more details.

(C3) $W^u(\gamma_*)$ and $W^s(x_-)$, as well as $W^s(\gamma_*)$ and $W^u(x_+)$, intersect transversely, where $W^u(\gamma_*)$ and $W^s(\gamma_*)$ stand for the stable and unstable sets of the non-hyperbolic periodic orbit γ_*.

The name *cusp-transverse* comes from the fact that under the genericity condition (C2) from results in [6] (cited as Theorem 2.4 in [5]) it follows that, the Poincaré map along the saddle-node periodic orbit has a fixed point whose stable and unstable sets form a cusp (see Figure 1). Condition (C3) says that they intersect transversely stable and unstable manifolds of equilibrium points $x_\pm$. Under the bifurcation, the saddle-node periodic orbit will split into two periodic orbits for λ on the one side of λ_*, while no periodic orbit will present near γ_* for λ on the other side of λ_*.The cocooning cascade occurs for the latter values of λ.

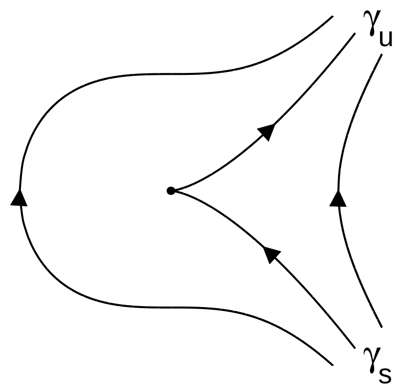

Figure 1: The stable and unstable set of the saddle-node periodic orbit on the Poincaré section [5]

The following theorem was proved in [5, Theorem 1.5]

Theorem 1 *Let X_λ ba a smooth family of reversible vector fields on $\mathbb{R}^3$ with (H1) and (H2). Suppose that at $\lambda = \lambda_0$ the corresponding vector field X_{λ_0} has a cusp-transverse heteroclinic chain. Then the family exhibits a cocooning cascade of heteroclinic tangencies centered at λ_0.*

In order to apply this theorem to the Michelson system (1), one has to verify the presence of the cusp-transverse heteroclinic chain including the conditions (C1)-(C3). All but (C3) are relatively easy to check by interval arithmetic, but it is not straightforward to verify the transversality in (C3), because the stable and unstable sets are associated to a non-hyperbolic periodic orbit and therefore a standard technique for rigorous enclosure of these manifolds does not work. Nevertheless, we have succeeded in proving the existence of a cusp-transverse heteroclinic chain in the Michelson system, and hence we have obtained a computer assisted proof of the following fact:

Theorem 2 *There exists* $c_\infty \in [1.2662323370670545, 1.2662323370713253]$ *(compare Theorem 12), such that Michelson system (1) exhibits a cocooning cascade of heteroclinic tangencies centered at* c_∞.

The proof of this theorem is based on the ideas from the proof of Theorem 1, but replaces the condition (C3) by its topological version:

(C3t) $W^u(\gamma_*)$ and $W^s(x_-)$, as well as $W^s(\gamma_*)$ and $W^u(x_+)$, intersect. Moreover, these intersections are topologically transverse.

Since the main idea of proving Theorem 1 in [5, Theorem 1.5] uses topological arguments like the intermediate value theorem, it is obvious to see that all arguments used in the proof are also valid, even if the condition (C3) is replaced by (C3t). Therefore, once we verify the weaker condition (C3t), we can conclude the cocooning cascade of heteroclinic tangencies centered at c_∞.

In this paper, we use rigorous numerical methods developed by CAPD group [2, 21, 22] to verify the conditions (C1,C2,C3t).

While this paper focuses on the proof the existence of the cocoon bifurcation in a concrete system, the approach used in this paper is general and can be without any changes applied to other systems with a similar bifurcation. A new result in this context is the existence of a Lyapunov function at the bifurcation point - see Lemma 16 (after we finished our paper we learned that this fact in a context of behavior of stable and unstable curves of the degenerate fixed point has been also established by Fontich in [7], see also the short discussion at the beginning of Section 9.1). Another new aspect is the introduction of the notion of the h-set with cones, which allows to link a topological tool, the covering relation, with cone conditions, a standard object in the smooth hyperbolic dynamics. This is used to rigorously estimate the stable and unstable manifolds of fixed points or periodic orbits.

2 Notation

2.1 Intervals

Frequently, when discussing computer assisted proofs of some facts we will use the notion of an *interval hull* of some object, which will be always denoted by $[A]$, where A is the mathematical object. Below we give a precise definition of the interval hull for the objects we are using in present work.

Definition 3 Let $h : \mathbb{R}^n \to \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2 map, $g : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map and $U \subset \mathbb{R}^n$. We define

$$\begin{aligned}
[h(U)] &= [\inf_{x\in U} h(x), \sup_{x\in U} h(x)] \\
[Dg(U)] &= \left\{ A \in \mathbb{R}^{n\times n} \mid \forall_{ij} A_{ij} \in \left[\frac{\partial g_i}{\partial x_j}(U) \right] \right\} \\
\left[D^2 f(U)\right] &= \left\{ A \in \mathbb{R}^{n\times n} \mid A = A^T,\ \forall_{ij} A_{ij} \in \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(U) \right] \right\}
\end{aligned}$$

2.2 Other notation

For a map $R : X \to X$, its fixed point set is denoted by $\mathrm{Fix}(R) = \{x \in X \mid R(x) = x\}$.

In $\mathbb{R}^n$, unless it is stated otherwise, we will use the euclidian norm $\|x\| = \sqrt{\sum_i x_i^2}$ to measure distances. Let $x_0 \in \mathbb{R}^s$, then $B_s(x_0, r) = \{z \in \mathbb{R}^2 \mid \|x_0 - z\| < r\}$ and $B_s = B(0, 1)$.

We will often consider maps and points in $\mathbb{R}^u \times \mathbb{R}^s$. For $z \in \mathbb{R}^u \times \mathbb{R}^s$ we will usually call x the first coordinate, and y the second one. Hence $z = (x, y)$, where $x \in \mathbb{R}^u$ and $y \in \mathbb{R}^s$. We will also use the projection maps $\pi_1(z) = \pi_x(z) = x$ and $\pi_2(z) = \pi_y(z) = y$.

Let $z \in \mathbb{R}^n$ and $U \subset \mathbb{R}^n$ be a compact set and $f : U \to \mathbb{R}^n$ be a continuous map, such that $z \notin f(\partial U)$. Then the local Brouwer degree [16] of f on U at z is defined and will be denoted by $\deg(f, U, z)$.

By $f : U \multimap W$ we will denote a partial (or local map), which means that the domain of f is contained in U.

3 Covering relations, horizontal and vertical disks

Notions of covering relation, horizontal and vertical disks between sets are the main topological tool used to establish the existence of topologically transverse intersections. For the convenience of the reader we will recall here their definitions and basic theorems about them.

Definition 4 [8, Definition 1] An *h-set*, N, is a quadruple $(|N|, u(N), s(N), c_N)$ such that

- $|N|$ is a compact subset of $\mathbb{R}^n$
- $u(N), s(N) \in \{0, 1, 2, \dots\}$ are such that $u(N) + s(N) = n$
- $c_N : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$ is a homeomorphism such that

$$c_N(|N|) = \overline{B_{u(N)}} \times \overline{B_{s(N)}}.$$

We set

$$\begin{aligned} \dim(N) &:= n, \\ N_c &:= \overline{B_{u(N)}} \times \overline{B_{s(N)}}, \\ N_c^- &:= \partial B_{u(N)} \times \overline{B_{s(N)}}, \\ N_c^+ &:= \overline{B_{u(N)}} \times \partial B_{s(N)}, \\ N^- := c_N^{-1}(N_c^-), \quad N^+ &= c_N^{-1}(N_c^+). \end{aligned}$$

Hence an h-set N is a product of two closed balls in some coordinate system. The numbers $u(N)$ and $s(N)$ are called the nominally unstable and nominally stable dimensions, respectively. The subscript c refers to the new coordinates given by homeomorphism c_N. Observe that if $u(N) = 0$, then $N^- = \emptyset$ and if $s(N) = 0$, then $N^+ = \emptyset$.

Definition 5 [19, Definition 2.2] Assume that N, M are h-sets, such that $u(N) = u(M) = u$ and let $f : N \to \mathbb{R}^{\dim(M)}$ be continuous. Let $f_c = c_M \circ f \circ c_N^{-1} : N_c \to \mathbb{R}^u \times \mathbb{R}^{s(M)}$. We say that N *f-covers* M, denoted by

$$N \overset{f}{\Longrightarrow} M,$$

if the following two conditions are satisfied

1. there exists a homotopy $h : [0,1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^{s(M)}$ such that

$$\begin{aligned} h_0 &= f_c, && (4)\\ h([0,1], N_c^-) \cap M_c &= \emptyset, && (5)\\ h([0,1], N_c) \cap M_c^+ &= \emptyset. && (6) \end{aligned}$$

2. There exists a linear map $A : \mathbb{R}^u \to \mathbb{R}^u$ such that

$$\begin{aligned} h_1(p,q) &= (A(p), 0), \text{ for } p \in \overline{B_u} \text{ and } q \in \overline{B_{s(N)}}, && (7)\\ A(\partial B_u) &\subset \mathbb{R}^u \setminus \overline{B_u}. && (8) \end{aligned}$$

Observe that in the above definition $s(N)$ and $s(M)$ can be different.

Definition 6 [20, Definition 10] Let N be an h-set. Let $b : \overline{B_{u(N)}} \to |N|$ be continuous and let $b_c = c_N \circ b$. We say that b is *a horizontal disk in* N if there exists a homotopy $h : [0,1] \times \overline{B_{u(N)}} \to N_c$, such that

$$\begin{aligned} h_0 &= b_c && (9)\\ h_1(x) &= (x, 0), \qquad \text{for all } x \in \overline{B_{u(N)}} && (10)\\ h(t,x) &\in N_c^-, \qquad \text{for all } t \in [0,1] \text{ and } x \in \partial B_{u(N)} && (11) \end{aligned}$$

We will say that b is *a proper horizontal disk in* N if it is a horizontal disk in N and $b(\overline{B_{u(N)}}) \cap N^+ = \emptyset$.

Definition 7 [20, Definition 11] Let N be an h-set. Let $b : \overline{B_{s(N)}} \to |N|$ be continuous and let $b_c = c_N \circ b$. We say that b is *a vertical disk in* N if there exists a homotopy $h : [0,1] \times \overline{B_{s(N)}} \to N_c$, such that

$$\begin{aligned} h_0 &= b_c \\ h_1(x) &= (0, x), \qquad \text{for all } x \in \overline{B_{s(N)}} \\ h(t,x) &\in N_c^+, \qquad \text{for all } t \in [0,1] \text{ and } x \in \partial B_{s(N)}. && (12) \end{aligned}$$

We will say that b is *a proper vertical disk in* N if it is a vertical disk in N and $b(\overline{B_{s(N)}}) \cap N^- = \emptyset$.

Definition 8 Let N be an h-set in $\mathbb{R}^n$ and b be a horizontal (vertical) disk in N. We will say that $x \in \mathbb{R}^n$ *belongs to* b, when $b(z) = x$ for some $z \in \mathrm{dom}(b)$.

Let $Z \subset \mathbb{R}^n$. We will say that Z *contains disk* b if for all $x \in \mathrm{dom}(b)$ holds $b(x) \in Z$.

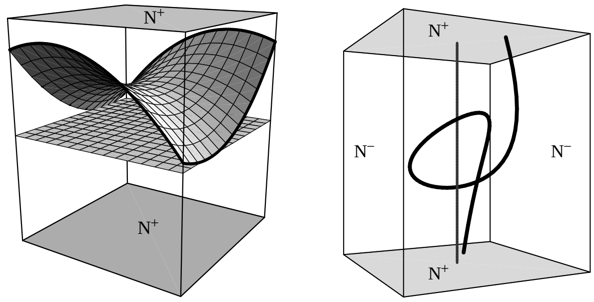

Figure 2: A horizontal disk in an h-set N with $u(N) = 2$ and $s(N) = 1$ (left). A vertical disk in an h-set N with $u(N) = 2$ and $s(N) = 1$ (right).

3.1 Representation of h-sets

In the present paper we will use only h-sets possessing exactly one unstable direction. Therefore we use the following representation. An h-set N in $\mathbb{R}^n$ may be defined by specifying a sequence $(x, u, s_1, \ldots, s_{n-1})$, where $x, u, s_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, n-1$ are such that $u, s_1, \ldots, s_{n-1}$ are linearly independent. We then set

$$\begin{aligned} |N| &= \{v \in \mathbb{R}^n \mid \exists_{t_1,t_2,\ldots,t_n \in [-1,1]} \quad v = x + t_1 s_1 + \cdots + t_{n-1}s_{n-1} + t_n u\} \\ &= x + [-1,1] \cdot u + [-1,1] \cdot s_1 + \cdots + [-1,1] \cdot s_{n-1}. \end{aligned}$$

and take u as the nominally unstable direction and s_i as the nominally stable directions. The homeomorphism c_N is taken as the affine map $c_N(v) = M^{-1}(v - x)$, where $M = [u^T, s_1^T, \ldots, s_{n-1}^T]$ is a square matrix. In this representation $N_c = \overline{B_1} \times \overline{B_{n-1}} = [-1,1]^n$ is a product of unit balls in the maximum norm.

In such a situation we will write $N = \mathfrak{h}(x, u, s_1, s_2, \ldots, s_{n-1})$.

Given an h-set $N = \mathfrak{h}(c, u, s)$ on the plane by N^r, N^l, N^t and N^b we denote the right, left, top and bottom edge of $|N|$, respectively. More precisely,

$$\begin{aligned} N^r &= c + u + [-1,1] \cdot s \\ N^l &= c - u + [-1,1] \cdot s \\ N^t &= c + [-1,1] \cdot u + s \\ N^b &= c + [-1,1] \cdot u - s \end{aligned}$$

4 Topologically transverse intersections

Definition 9 Assume that M_1 and M_2 are manifolds immersed in $\mathbb{R}^n$, such that $\dim(M_1) = u$ and $\dim(M_2) = s$, $s + u = n$.

Assume that N is an h-set in $\mathbb{R}^n$, such that $u(N) = u$ and $s(N) = s$.

We will say that M_1 *and* M_2 *have topologically transverse intersection in* N, if the following conditions are satisfied:

- there exists a proper horizontal disk b_1 in N that is contained in M_1; and
- there exists a proper vertical disk b_2 in N that is contained in M_2.

Theorem 3 *Assume that* M_1 *and* M_2 *have topologically transverse intersection in* N*, then there exist a* $p \in M_1 \cap M_2 \cap \operatorname{int} N$.

Proof: The assertion follows immediately from [20, Thm. 4] applied to the chain $N \overset{Id}{\Longrightarrow} N$. ∎

5 Cone conditions

The goal of this section is to introduce a method, which will allow to handle relatively easily the hyperbolic structure on h-sets.

5.1 Horizontal and vertical disks satisfying cone conditions

Definition 10 Let $N \subset \mathbb{R}^n$ be an h-set and $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form

$$Q(x, y) = \alpha(x) - \beta(y), \qquad (x, y) \in \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)},$$

where $\alpha : \mathbb{R}^{u(N)} \to \mathbb{R}$, and $\beta : \mathbb{R}^{s(N)} \to \mathbb{R}$ are positively definite quadratic forms.

The pair (N, Q) will be called an *h-set with cones.*

Quite often we will drop Q in the symbol (N, Q) and we will say that N is an h-set with cones.

Definition 11 Let (N, Q) be an h-set with cones and let $b : \overline{B_u} \to |N|$ be a horizontal disk. We will say that b *satisfies the cone condition (with respect to* Q*)*, if any $x_1, x_2 \in \overline{B_u}$ with $x_1 \neq x_2$ satisfy

$$Q(b_c(x_1) - b_c(x_2)) > 0.$$

Definition 12 Let (N, Q) be an h-set with cones and let $b : \overline{B_s} \to |N|$ be a vertical disk. We will say that b *satisfies the cone condition (with respect to* Q*)*, if any $y_1, y_2 \in \overline{B_s}$ with $y_1 \neq y_2$ satisfy

$$Q(b_c(y_1) - b_c(y_2)) < 0.$$

The following theorem says that horizontal and vertical disks satisfying cone conditions are graphs of Lipschitz functions.

Theorem 4 *Let (N, Q) be a h-set with cones and let $b : \overline{B_u} \to |N|$ be a horizontal disk satisfying the cone condition.*

Then there exists a Lipschitz function $y : \overline{B_u} \to \overline{B_s}$ such that

$$b_c(x) = (x, y(x)). \tag{13}$$

Analogously, if $b : \overline{B_s} \to |N|$ is a vertical disk satisfying the cone condition, then there exists a Lipschitz function $x : \overline{B_s} \to \overline{B_u}$

$$b_c(y) = (x(y), y)). \tag{14}$$

Proof: We will prove only the first part, the proof of the other part is analogous.

In the first part of this proof we will show that for any $x \in \text{int } B_{u(N)}$ there exists $y_x \in \overline{B_{s(N)}}$, such that

$$b_c(z) = (x, y_x), \qquad \text{for some } z \in N_c. \tag{15}$$

For this we will use the local Brouwer degree

In the second part using the cone condition we will show that y_x is uniquely defined and its dependence on x is Lipschitz. Then we extend the definition of $y(x)$ to $x \in \partial B_u$.

To prove (15) consider a homotopy $\pi_1 \circ h : [0, 1] \times \overline{B_{u(N)}} \to \overline{B_{u(N)}}$. Let $x \in \text{int } B_{u(N)}$, where $\pi_1 : \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)} \to \mathbb{R}^{u(N)}$ is a projection on the first component. It is easy to see that

$$\deg(\pi_1 \circ b_c, \overline{B_{u(N)}}, x) = \deg(\pi_1 \circ h_1, \overline{B_{u(N)}}, x) = \deg(\text{Id}, \overline{B_{u(N)}}, x) = 1. \tag{16}$$

This proves (15).

To prove the uniqueness of y_x, assume that $y_1, y_2 \in \overline{B_{s(N)}}$, $y_1 \neq y_2$ be such that

$$b_c(z_1) = (x, y_1), \qquad b_c(z_2) = (x, y_2). \tag{17}$$

From the cone condition for b it follows that

$$0 < Q(b_c(z_1) - b_c(z_2)) = \alpha(0) - \beta(y_1 - y_2) < 0 \tag{18}$$

which is a contradiction. Hence we have a well defined function

$$y(x) = y_x, \qquad \text{for } x \in \text{int } B_{u(N)}. \tag{19}$$

Observe that from the cone condition it follows that for any $x_1, x_2 \in \text{int } B_{u(N)}$, $x_1 \neq x_2$ holds

$$A\|x_1 - x_2\|^2 \geq \alpha(x_1 - x_2) > \beta(y(x_1) - y(x_2)) \geq C\|y(x_1) - y(x_2)\|^2, \tag{20}$$

where $A > 0, C > 0$ are some real constants.

This proves the Lipschitz condition. It is easy to see that the function $y(x)$ can be extended also to the boundary of $B_{u(N)}$. ∎

5.2 The link between covering relations and cone conditions

Definition 13 Assume that $(N, Q_N), (M, Q_M)$ are h-sets with cones, such that $u(N) = u(M) = u$ and let $f : N \to \mathbb{R}^{\dim(M)}$ be continuous. Assume that $N \overset{f}{\Longrightarrow} M$. We say that f satisfies the cone condition (with respect to the pair (N, M)), if any $x_1, x_2 \in N_c$ with $x_1 \neq x_2$ satisfy

$$Q_M(f_c(x_1) - f_c(x_2)) > Q_N(x_1 - x_2).$$

Whenever it is convenient, we will also say that the cone conditions are satisfied for the covering relation $N \overset{f}{\Longrightarrow} M$, if the above condition is satisfied.

The basic theorem linking covering relation and cone conditions is:

Theorem 5 *Assume that*

$$N_0 \overset{f_0}{\Longrightarrow} N_1 \overset{f_1}{\Longrightarrow} N_2 \overset{f_2}{\Longrightarrow} \cdots \overset{f_{k-1}}{\Longrightarrow} N_k,$$

where all h-sets are h-sets with cones and f_i $(i = 0, \ldots, k-1)$ *satisfies the cone condition with respect to pair* (N_i, N_{i+1}) *. Assume that* $b : \overline{B_{s(N_k)}} \to |N_k|$ *is a vertical disk in* N_k *satisfying the cone condition.*

Then, in N_0*, there exists a vertical disk* $b_0 : \overline{B_{s(N_0)}} \to |N_0|$ *satisfying the cone condition and such that for all* $y \in \overline{B_{s(N_0)}}$ *holds*

$$\begin{aligned} f_{i-1} \circ f_{i-2} \circ \cdots \circ f_0(b_0(y)) &\in N_i, \qquad \textit{for } i = 1, \ldots, k \\ f_{k-1} \circ \cdots \circ f_0(b_0(y)) &= b_k(y_1), \qquad \textit{for some } y_1 \in \overline{B_{s(N_k)}} \end{aligned}$$

Proof: For the proof it is enough to consider only the case of $k = 1$. For $k > 1$ the result follows by induction.

Without loss of generality we can assume that $N_0 = N_{0,c} = \overline{B_{u(N_0)}} \times \overline{B_{s(N_0)}}$, $N_1 = N_{1,c} = \overline{B_{u(N_1)}} \times \overline{B_{s(N_1)}}$, $f_0 = f_{0,c}$. Consider a family of horizontal disks in N_0 $d_y : \overline{B_{u(N_0)}} \to N_0$ for $y \in \overline{B_{s(N_0)}}$

$$d_y(x) = (x, y).$$

From Theorem 4 in [20] it follows that for each $y \in \overline{B_{s(N_0)}}$ there exists $x \in \overline{B_{u(N_0)}}$, such that

$$f_0(x, y) = b(y_1), \qquad \text{for some } y_1 \in \overline{B_{s(N_1)}}. \tag{21}$$

Let us fix $y \in \overline{B_{s(N_0)}}$. We will show that there exists only one x satisfying (21). For the proof assume the contrary, hence we have $x_1 \neq x_2$ and x_1, x_2 both satisfy (21).

Observe that $Q_{N_0}((x_1, y) - (x_2, y)) > 0$, hence from the fact that f_0 satisfies the cone condition it follows that

$$Q_{N_1}(f_0(x_1, y) - f_0(x_2, y))) > 0.$$

But the above condition is in a contradiction with the definition of a vertical disk satisfying the cone condition. Hence (21) defines a function $x(y)$ in a unique way.

It is easy to see that this function is continuous. For the proof from the compactness argument it follows that it is enough to prove that if we have a sequence of pairs (x_n, y_n), where $y_n \in \overline{B_s}$, $y_n \to \bar{y}$ for $n \to \infty$ and $x_n = x(y_n)$, $x_n \to \bar{x}$, then $f_0(\bar{x}, \bar{y}) \in b(\overline{B_{s(N_1)}})$, but this is an obvious consequence of the continuity of f_0 and the compactness of $b(\overline{B_{s(N_1)}})$.

It is easy to see that $b_0(y) = (x(y), y)$ is a vertical disk in N_0. It remains to show that it satisfies the cone condition.

We will prove this by a contradiction. Assume that we have y_1 and y_2 such that

$$Q_{N_0}((x(y_1), y_1) - (x(y_2), y_2)) \geq 0,$$

then

$$Q_{N_1}(f_0(x(y_1), y_1) - f_0(x(y_2), y_2)) > 0,$$

hence the points $f_0(x(y_1), y_1)$ and $f_0(x(y_2), y_2)$ both cannot belong to b_1, because otherwise the cone condition is violated. ∎

5.3 The verification of cone conditions

Assume that (N, Q_N) and (M, Q_M) are h-sets with cones and a map $f : N \to \mathbb{R}^{\dim(M)}$ is C^1.

Our intention is to give a condition, which will guarantee that N f-covers M and it satisfies the cone conditions.

Let $[Df_c(N_c)]_I$ be the interval enclosure of Df_c on N_c. Observe that when $\dim(M) \neq \dim(N)$ this is not a square matrix.

Lemma 6 *Assume that for any $B \in [Df_c(N_c)]$, the quadratic form*

$$V(x) = Q_M(Bx) - Q_N(x)$$

is positively definite, then for any $x_1, x_2 \in N_c$ such that $x_1 \neq x_2$ holds

$$Q_M(f_c(x_1) - f_c(x_2)) > Q_N(x_1 - x_2).$$

Proof: Let us fix x_1, x_2 in N_c. We have

$$f_c(x_2) - f_c(x_1) = \int_0^1 Df_c(x_1 + t(x_2 - x_1))dt \cdot (x_2 - x_1).$$

Let $B = \int_0^1 Df_c(x_1 + t(x_2 - x_1))dt$. Obviously $B \in [Df_c]_I$. Hence

$$f_c(x_2) - f_c(x_1) = B(x_2 - x_1).$$

We have

$$\begin{aligned} & Q_M(f_c(x_2) - f_c(x_1)) - Q_N(x_2 - x_1) \\ = \; & Q_M(B(x_2 - x_1)) - Q_N(x_2 - x_1) \\ = \; & V(x_2 - x_1) > 0. \end{aligned}$$

∎

In the light of the above lemma the verification of cone conditions can be reduced to checking that the interval matrix corresponding to the quadratic form V for various choices of $B \in [Df_c(N_c)]_I$ given by

$$V = [Df_c(N_c)]_I^T Q_M [Df_c(N_c)]_I - Q_N$$

is positively definite.

6 The invariant manifolds of hyperbolic fixed points, covering relations and cone conditions

6.1 Maps

Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ be a local diffeomorphism (hence at least C^1).

Assume that $z_0 = (x_0, y_0)$ is an hyperbolic fixed point for f. This by the definition means that all the eigenvalues of $Df(z_0)$ do not belong to the unit circle.

Let $Z \subset \mathbb{R}^n$, $x_0 \in Z$, $Z \subset \text{dom}(f)$. We define

$$\begin{aligned}
W^s_Z(f, z_0) &= \{z \mid \forall_{n \geq 0} f^n(z) \in Z, \quad \lim_{n\to\infty} f^n(z) = z_0\} \\
W^u_Z(f, z_0) &= \{z \mid \forall_{n \leq 0} f^n(z) \in Z, \quad \lim_{n\to-\infty} f^n(z) = z_0\} \\
W^s(f, z_0) &= \{z \mid \lim_{n\to\infty} f^n(z) = z_0\} \\
W^u(f, z_0) &= \{z \mid \lim_{n\to-\infty} f^n(x) = z_0\} \\
Inv^+(f, Z) &= \{z \mid \forall_{n \geq 0} f^n(z) \in Z\} \\
Inv^-(f, Z) &= \{z \mid \forall_{n \leq 0} f^n(z) \in Z\}
\end{aligned}$$

If f is know from the context, then we will usually drop it and use $W^s(z_0)$, etc. instead.

Theorem 7 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a local diffeomorphism. Assume that z_0 is a hyperbolic fixed point of f, there exists an h-set N with cones, $z_0 \in N$,*

$$N \stackrel{f}{\Longrightarrow} N,$$

and f satisfies cone conditions with respect to the pair (N, N).

Then $W^s_N(z_0)$ is a vertical disk in N satisfying the cone condition.

Proof: Our first goal is to prove that

$$Inv^+ N = W^s_N(z_0). \tag{22}$$

To prove (22) it is enough to show that, if $f^n(z) \in N$ for all $n \geq \mathbb{N}$, then $\lim_{n\to\infty} f^n(z) = z_0$.

Observe that the function $V(z) = Q(z - z_0)$ is a Lyapunov function on N, i.e. it increases on non constant orbits in N. Hence $f^n(z)$ must for $n \to \infty$ converge toward the equilibrium. It is easy to see, by the Lyapunov function argument that there is only one fixed point in N. This finishes the proof of (22).

Now we show, that $W^s_N(z_0)$ is a vertical disk in N satisfying the cone condition

First we show that for all $y \in \overline{B_s}$ there exists $x \in \overline{B_u}$, such that

$$z = c_N^{-1}(x, y) \in W^s_N(x_0). \tag{23}$$

By condition (22) it is equivalent to showing that

$$f^n(z) \in N, \quad \text{for } n \in \mathbb{N}. \tag{24}$$

Consider a family of horizontal disks in N $d_y : \overline{B_{u(N)}} \to N$ for $y \in \overline{B_{s(N)}}$

$$d_y(x) = (x, y).$$

Consider an infinite chain of covering relations

$$N \overset{f}{\Longrightarrow} N \overset{f}{\Longrightarrow} N \overset{f}{\Longrightarrow} \cdots N \overset{f}{\Longrightarrow} \cdots \tag{25}$$

From [19, Corollary 3.10] applied to d_y and the chain (25) it follows that for every $y \in \overline{B_s}$ there exists $x \in \overline{B_u}$, such that (24) holds.

The next step is to prove that such x is unique. Let us assume the contrary, then there exists $y \in \overline{B_s}$ and $x_1, x_2 \in \overline{B_u}$, $x_1 \neq x_2$, such that $z_i = c_N^{-1}(x_i, y)$ for $i = 1, 2$ satisfy conditions (23,24). Observe that

$$Q(z_1 - z_2) = \alpha(x_1 - x_2) > 0,$$

hence from the cone condition and (24) it follows that

$$Q(f^n(z_1) - f^n(z_2) > \alpha(x_1 - x_2), \qquad \text{for } n \in \mathbb{N}.$$

Passing to the limit $n \to \infty$ we obtain

$$0 = Q(z_0 - z_0) = \lim_{n\to\infty} Q(z_1(t) - z_2(t)) > \alpha(x_1 - x_2) > 0.$$

This is a contradiction. Hence we have a well defined function $x(y)$ on $\overline{B_s}$.

From the uniqueness the continuity of $x(y)$ follows easily. Namely, from the compactness argument it follows that it is enough to prove that if we have a sequence of pairs (x_n, y_n), where $y_n \in \overline{B_s}$, $y_n \to \bar{y}$ for $n \to \infty$ and $x_n = x(y_n)$, $x_n \to \bar{x}$, then $c_N^{-1}(\bar{x}, \bar{y}) \in Inv^+ N$, but this is an obvious consequence of the closeness of $Inv^+ N$.

The last step - the cone condition. Till now we have proved that the set $W^s_N(x_0)$ forms a vertical disk in N, given by $b(x, y) = c_N^{-1}(x(y), y)$.

We have to check whether

$$Q_N((x(y_1), y_1) - (x(y_2), y_2)) < 0, \qquad \text{for all } y_1, y_2 \in \overline{B_s},\ y_1 \neq y_2 \tag{26}$$

Assume that it does not hold. Then for some $z_1, z_2 \in W^s_N(x_0)$, $z_1 \neq z_2$ we have

$$Q(z_1 - z_2) \geq 0.$$

From the cone condition it follows that

$$Q(f^n(z_1) - f^n(z_2) > Q(f(z_1) - f(z_2)) > 0, \qquad \text{for } n > 1.$$

Passing to the limit $n \to \infty$ we obtain

$$0 = Q(z_0 - z_0) = \lim_{n\to\infty} Q(f^n(z_1) - f^n(z_2)) > Q(f(z_1) - f(z_2)) > 0.$$

This is a contradiction, and hence proves (26). ∎

6.2 ODEs

Consider an ordinary differential equation

$$z' = f(z), \qquad z \in \mathbb{R}^n, \quad f \in C^1(\mathbb{R}^n, \mathbb{R}^n). \tag{27}$$

Let us denote by $\varphi(t, p)$ the solution of (27) with the initial condition $z(0) = p$.

Assume that z_0 is a hyperbolic fixed point for (27). This by the definition means that all the eigenvalues of $Df(z_0)$ have nonzero real part.

Let $Z \subset \mathbb{R}^n$, $x_0 \in Z$. We define

$$\begin{aligned}
W^s_Z(\varphi, z_0) &= \{z \mid \forall_{t\geq 0}\varphi(t,z) \in Z, \quad \lim_{t\to\infty} \varphi(t,z) = z_0\}\\
W^u_Z(\varphi, z_0) &= \{z \mid \forall_{t\leq 0}\varphi(t,z) \in Z, \quad \lim_{t\to-\infty} \varphi(t,z) = z_0\}\\
W^s(\varphi, z_0) &= \{z \mid \lim_{t\to\infty} \varphi(t,z) = z_0\}\\
W^u(\varphi, z_0) &= \{z \mid \lim_{t\to-\infty} \varphi(t,x) = z_0\}
\end{aligned}$$

Theorem 8 *Consider (27). Assume that z_0 is a hyperbolic fixed point of (27)*

Let $T > 0$ and let $\varphi_T = \varphi(T, \cdot)$. Assume that there exists an h-set N with cones, $z_0 \in N$,

$$N \overset{\varphi_T}{\Longrightarrow} N,$$

and φ_T satisfies cone conditions with respect to the pair (N, N).

Then $W^s(\varphi, z_0)$ contains a vertical disk in N satisfying the cone condition.

Proof: The assertion follows immediately from Theorem 7 and the obvious inclusion $W^s_N(\varphi_T, z_0) \subset W^s(\varphi, z_0)$. ∎

7 The existence of a saddle-node bifurcation point in the Michelson system

Our goal of this section is to prove the conditions (C1) and (C2) for the Michelson system (1).

Let us fix the section $\Theta = \{(x, y, 0) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$. On Θ we will use the coordinates (x, y). The vector field given by (1) is transverse to Θ except on the parabola $c^2 - y - \frac{1}{2}x^2 = 0$, and therefore as long as an orbit stay away from this parabola, we can consider the Poincaré return map P on subsets of Θ. The reader should be warned that we in fact consider the half-return map. Namely, we define a map P as the first return map to Θ, regardless the direction of the vector field, and hence, if at our starting point $p \in \Theta$ $z'(p) > 0$, then for $q = P(z)$ we have $z'(q) < 0$. Therefore the usual Poincaré return map Π corresponds to P^2 in this notation. In particular, every periodic orbit γ intersecting Θ corresponds to a fixed point of P^{2n}, where n is the number of intersections of γ with Θ in one period with $z' > 0$.

It is easy to see that P has the time-reversing symmetry

$$R \circ P^2 \circ R = (P^2)^{-1} \tag{28}$$

with respect to the involution on Θ given by

$$R(x, y) = (-x, y).$$

The fixed point set of R, $\mathrm{Fix}(R)$, is the line $x = 0$.

Now, the conditions (C1) and (C2) can be formulated precisely in terms of the Poincaré return map P^2 as follows:

(C1) The Poincaré map $\Pi = P^2$ has a saddle-node periodic point v_∞ of some period k on $\Theta \cap \mathrm{Fix}(R)$ at a parameter value $c = c_\infty$.

Let $Q = P^{2k} = \Pi^k$. Then from (C1), Q has a saddle-node fixed point v_∞ at $c = c_\infty$, and hence its linearization $DQ(v_\infty)$ has unity as an eigenvalue. Because Q is reversible and orientation preserving, the other eigenvalue is also unity. Therefore, unless $DQ(v_\infty)$ itself is the identity matrix, it is a unipotent matrix, namely it is linearly conjugate to

$$\begin{bmatrix} 1 & 0 \\ K & 1 \end{bmatrix}$$

with $K \neq 0$ (in fact, it is possible to choose $K = 1$). Assume we make coordinate change so that the resulting linearization matrix has the above form. Note that we still keep the same notation (x, y) for the new coordinates. (As a matter of fact, we shall see that, in our case, the linearization matrix takes the above form with respect to the original coordinates, and hence there is no need to make coordinate change. See Lemma 10.)

(C2) Under the reversibility (28), the saddle-node bifurcation takes place generically, namely, the map $Q = (Q_1, Q_2)$ (under the new coordinates as above) satisfies

$$\frac{\partial Q_1}{\partial c}(c_\infty, v_\infty) \neq 0, \qquad \frac{\partial^2 Q_1}{\partial y^2}(c_\infty, v_\infty) \neq 0. \tag{29}$$

7.1 Existence of the bifurcation point

In order to prove (C1) and (C2) we need to show that for a parameter value $c = c_\infty$ a symmetric periodic orbit is born in the saddle-node bifurcation.

The standard way to achieve this is to look at $P^k(\mathrm{Fix}(R)) \cap \mathrm{Fix}(R)$ for various values of the parameter c, because if $v \in P^k(\mathrm{Fix}(R)) \cap \mathrm{Fix}(R)$, then $P^{2k}(v) = v$.

Figure 3 presents the numerical evidence of the existence of the saddle-node bifurcation. For $c > c_\infty$ $P^2(\mathrm{Fix}(R))$ is to the left of $\mathrm{Fix}(R)$, at $c = c_\infty$ apparently we have a quadratic-like tangency of $P^2(\mathrm{Fix}(R))$ and $\mathrm{Fix}(R)$ - this is the bifurcation point - and for $c < c_\infty$ we have two intersection points corresponding to two symmetric periodic orbits.

Let $P = (P_1, P_2) : (0, \infty) \times \Theta \to \Theta$ be the Poincaré map for the system with a parameter value c.

With some abuse of notation we will also write $P^2 = (P_1^2, P_2^2)$, hence P_i^2 will be not a square of P_i, but the i-th component of P^2.

To find a bifurcation point satisfying conditions (C1), it is enough to solve the following system of equations

$$\begin{cases} P_1^2(c, (0, y)) & = 0 \\ \frac{\partial P_1^2}{\partial y}(c, (0, y)) & = 0 \end{cases} \tag{30}$$

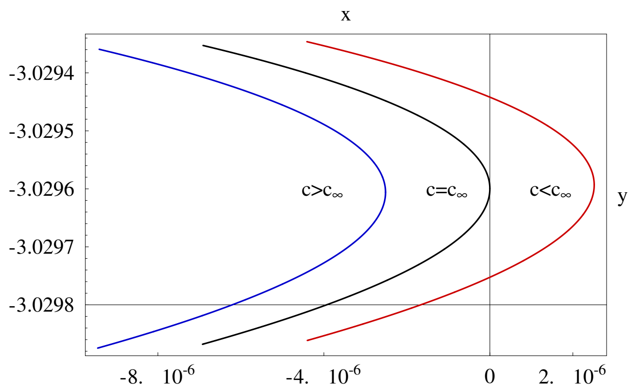

Figure 3: The image of a part of the y-axis under $P^2(c, \cdot)$ in the neighborhood of bifurcation point for certain parameter values.

As suggested by Figure 3, we expect system (30) to have a unique solution, hence it should be possible to prove it using the interval Newton method [1, 15] applied to the map

$$(c,y) \mapsto F(c,y) = \left(P_1^2(c,(0,y)), \frac{\partial P_1^2}{\partial y}(c,(0,y)) \right). \tag{31}$$

Put

$$\begin{aligned} C &= [1.2662323370670545, 1.2662323370713253] \\ Y &= [1.3591065061611036, 1.3591065061634906] \end{aligned} \tag{32}$$

Lemma 9 *The map F defined in (31) is smooth on $C \times Y$, and has a unique zero (c_∞, y_∞)in $C \times Y$. Moreover, the inequalities*

$$\frac{\partial^2 P_1^2}{\partial y^2}(c_\infty,(0,y_\infty)) \quad < \quad 0, \tag{33}$$

$$\frac{\partial P_1^2}{\partial c}(c_\infty,(0,y_\infty)) \quad < \quad 0 \tag{34}$$

are satisfied.

Proof: Let us denote by (c_0, y_0) the center of the rectangle $C \times Y$, let $X = C \times (\{0\} \times Y)$ and let $x_0 = (c_0,(0,y_0))$. The interval Newton operator for map F is given by ([1, 15]):

$$\begin{aligned} N(C\times Y) &= (c_0,y_0)^T - [DF(C\times Y)]^{-1} \cdot F(c_0,y_0) \\ &= \begin{bmatrix} c_0 \\ y_0 \end{bmatrix} - \begin{bmatrix} \frac{\partial P_1^2}{\partial c}(X) & \frac{\partial P_1^2}{\partial y}(X) \\ \frac{\partial^2 P_1^2}{\partial y \partial c}(X) & \frac{\partial^2 P_1^2}{\partial y^2}(X) \end{bmatrix}^{-1} \cdot \begin{bmatrix} P_1^2(x_0) \\ \frac{\partial P_1^2}{\partial y}(x_0) \end{bmatrix}. \end{aligned} \tag{35}$$

We need check whether $N(C \times Y) \subset \operatorname{int} C \times Y$.

In order to compute the partial derivatives that appear in (35), we use the $\mathcal{C}^2$-Lohner algorithm [22] applied to the system consisting of the equations (1) plus the equation $\dot{c} = 0$. Observe that this computation gives us bounds for $\frac{\partial P_1^2}{\partial y^2}(c,(0,y))$ and $\frac{\partial P_1^2}{\partial y}(c,(0,y))$ appearing in the conditions (33) and (34).

Let us define the Poincaré section for this system by $\widetilde{\Theta} := (0,\infty) \times \Theta$ and the Poincaré map $\widetilde{\Pi} : \widetilde{\Theta} \to \widetilde{\Theta}$ by

$$\widetilde{\Pi}(c,(x,y)) = (c, P^2(c,(x,y)))$$

We insert the whole set X as an initial condition in our routine computing the Poincaré map $\widetilde{\Pi}$ and its partial derivatives. With a computer assistance we showed that $\widetilde{\Pi}$ is well-defined and smooth on X. Moreover,

$$[DF(Y \times C)] = \begin{bmatrix} \left[\frac{\partial P_1^2}{\partial c}(X)\right] & \left[\frac{\partial P_1^2}{\partial y}(X)\right] \\ \left[\frac{\partial^2 P_1^2}{\partial y \partial c}(X)\right] & \left[\frac{\partial^2 P_1^2}{\partial y^2}(X)\right] \end{bmatrix},$$

where

$$\begin{aligned}
\left[\frac{\partial P_1^2}{\partial y}(X)\right] &\subset \quad [-1.3000889254044523, 1.2564083107236002]\cdot 10^{-10},\\
\left[\frac{\partial P_1^2}{\partial c}(X)\right] &\subset \quad [-2.5142004837175844, -2.514200482955935],\\
\left[\frac{\partial^2 P_1^2}{\partial y \partial c}(X)\right] &\subset \quad [5.780806228938423, 5.7808062332808534],\\
\left[\frac{\partial^2 P_1^2}{\partial y^2}(X)\right] &\subset \quad [-3.4588312295127772, -3.4588312278117295]
\end{aligned}$$

and

$$[N(C\times Y)] \subset \begin{bmatrix}[1.2662323370671162, 1.2662323370712558]\\ [1.3591065061621639, 1.3591065061624312]\end{bmatrix} \subset \operatorname{int}\,(C\times Y).$$

This proves that F has a unique zero (c_∞, y_∞) in $C \times Y$. Finally, we observe that the conditions (33) and (34) hold:

$$\begin{aligned}
\frac{\partial^2 P_1^2}{\partial y^2}(c_\infty,(0,y_\infty)) \in \left[\frac{\partial^2 P_1^2}{\partial y^2}(X)\right] < 0,\\
\frac{\partial P_1^2}{\partial c}(c_\infty,(0,y_\infty)) \in \left[\frac{\partial P_1^2}{\partial c}(X)\right] < 0.
\end{aligned}$$

∎

7.2 The form of derivatives of the Poincaré map at the bifurcation point

From the previous subsection, we see that $v_\infty = (0, y_\infty)$ is a saddle-node bifurcation point of $Q(x,y) = P^4(c,(x,y))$ at $c = c_\infty$. Let

$$DQ(0,y_\infty) = \begin{bmatrix} a & b\\ c & d\end{bmatrix}.$$

The reversibility implies $(R\circ DQ)^2 = \mathrm{Id}$, and hence

$$\mathrm{Id} = (R\circ DQ)^2 = \begin{bmatrix} a^2 - bc & b(a-d)\\ c(d-a) & d^2 - bc\end{bmatrix}.$$

This shows that $DQ(0,y_\infty)$ has one of the following forms:

$$\begin{bmatrix} a & b\\ \frac{a^2-1}{b} & a\end{bmatrix},\quad a\text{: arbitrary},\quad b\neq 0 \tag{36}$$

$$\begin{bmatrix} a & \frac{a^2-1}{c}\\ c & a\end{bmatrix},\quad a\text{: arbitrary},\quad c\neq 0 \tag{37}$$

$$\begin{bmatrix} \pm 1 & 0\\ 0 & \pm 1\end{bmatrix}. \tag{38}$$

Lemma 10 *Consider the saddle-node bifurcation point* (c_∞, v_∞)*, where* $v_\infty = (0, y_\infty)$*. Then we have*

$$DQ(v_\infty) = DP^4(c_\infty, v_\infty) = \begin{bmatrix} 1 & 0 \\ K & 1 \end{bmatrix} \tag{39}$$

with some $K > 0$*.*

Proof: As said before, $DQ(v_\infty)$ has one of the forms (36)-(38) as above, and both of its eigenvalues have to be equal to 1. We will show that, at the bifurcation point, $\frac{\partial P_1^4}{\partial y}(c_\infty, v_\infty) = 0$. From (30), we have $\frac{\partial P_1^2}{\partial y}(c_\infty, v_\infty) = 0$ and therefore

$$\begin{aligned} \frac{\partial P_1^4}{\partial y}(c_\infty, v_\infty) &= \frac{\partial P_1^2}{\partial x}(c_\infty, w)\frac{\partial P_1^2}{\partial y}(c_\infty, v_\infty) + \frac{\partial P_1^2}{\partial y}(c_\infty, w)\frac{\partial P_2^2}{\partial y}(c_\infty, v_\infty) \\ &= \frac{\partial P_1^2}{\partial y}(c_\infty, w)\frac{\partial P_2^2}{\partial y}(c_\infty, v_\infty), \end{aligned}$$

where $w = P^2(c_\infty, v_\infty)$. On the other hand, because of the reversibility, we have

$$R \circ P^2(c_\infty, \cdot) \circ R \circ P^2(c_\infty, \cdot) = \mathrm{Id},$$

wherever the left side is well defined. Hence

$$-P_1^2\left(c_\infty, \left(-P_1^2(c_\infty, (x, y)), P_2^2(c_\infty, (x, y))\right)\right) = x$$

holds identically. After taking the partial derivative of the above with respect to y and evaluating it at $(x, y) = (0, v_\infty)$, we obtain

$$-\frac{\partial P_1^2}{\partial x}(c_\infty, w)\frac{\partial P_1^2}{\partial y}(c_\infty, v_\infty) + \frac{\partial P_1^2}{\partial y}(c_\infty, w)\frac{\partial P_2^2}{\partial y}(c_\infty, v_\infty) = 0.$$

From this and (30), we get

$$\frac{\partial P_1^2}{\partial y}(c_\infty, w)\frac{\partial P_2^2}{\partial y}(c_\infty, v_\infty) = 0,$$

which shows that

$$\frac{\partial P_1^4}{\partial y}(c_\infty, v_\infty) = \frac{\partial P_1^2}{\partial y}(c_\infty, w)\frac{\partial P_2^2}{\partial y}(c_\infty, v_\infty) = 0,$$

and hence, $DQ(v_\infty) = DP^4(c_\infty, (0, y_\infty))$ takes either of the form (37) with $a = 1$, or (38).

Now, we can verify the following with computer assistance:

$$DP^4(c_\infty, (0, y_\infty)) \in \begin{bmatrix} \begin{bmatrix} \frac{\partial P_1^4}{\partial x}(X) \\ \frac{\partial P_2^4}{\partial x}(X) \end{bmatrix} & \begin{bmatrix} \frac{\partial P_1^4}{\partial y}(X) \\ \frac{\partial P_2^4}{\partial y}(X) \end{bmatrix} \end{bmatrix},$$

where $X = C \times (\{0\} \times Y)$ (see (32) for C and Y) and

$$\begin{aligned} \left[\frac{\partial P_1^4}{\partial x}(X)\right] &\subset [0.99999969282877466, 1.000000304317707] \\ \left[\frac{\partial P_1^4}{\partial y}(X)\right] &\subset [-6.595174451007324, 6.5661458936716599] \cdot 10^{-8} \\ \left[\frac{\partial P_2^4}{\partial x}(X)\right] &\subset [9.9806800319597109, 9.9806800978892678] \\ \left[\frac{\partial P_2^4}{\partial y}(X)\right] &\subset [0.99999998913436772, 1.0000000106169058] \end{aligned} \tag{40}$$

This clearly shows that $DP^4(c_\infty, v_\infty)$ has the form

$$DP^4(c_\infty, (0, y_\infty)) = \begin{bmatrix} 1 & 0 \\ K & 1 \end{bmatrix} \tag{41}$$

with $K \neq 0$. ∎

7.3 Genericity of the saddle-node bifurcation

We shall show that the conditions (33) and (34) are sufficient for the genericity of the saddle-node bifurcation.

Lemma 11 *Under the above circumstances, the conditions (33) and (34) imply the genericity condition (C2).*

Proof: From Lemma 9, we know that the map $Q = P^4$ has a saddle-node fixed point $v_\infty = (0, y_\infty)$ at $c = c_\infty$. Lemma 10 then shows that $DP^4(c_\infty, v_\infty)$ itself takes of the form

$$\begin{bmatrix} 1 & 0 \\ K & 1 \end{bmatrix}$$

for some $K \neq 0$, and hence the condition (29) in (C2) in this case read

$$\frac{\partial P_1^4}{\partial c}(c_\infty, v_\infty) \neq 0, \qquad \frac{\partial^2 P_1^4}{\partial y^2}(c_\infty, v_\infty) \neq 0, \tag{42}$$

in terms of the original coordinates (x, y) on Θ.

A simple calculation shows that

$$\begin{aligned} \frac{\partial}{\partial c} P^4(c, (x, y)) &= \frac{\partial P^2}{\partial c}(c, P^2(c, (x, y))) \\ &\quad + DP^2(c, P^2(c, (x, y))) \cdot \frac{\partial P^2}{\partial c}(c, (x, y)), \end{aligned}$$

and hence, recalling $w = P^2(c_\infty, v_\infty)$,

$$\frac{\partial}{\partial c} P^4(c_\infty, v_\infty) = \frac{\partial P^2}{\partial c}(c_\infty, w) + DP^2(c_\infty, w) \cdot \frac{\partial P^2}{\partial c}(c_\infty, v_\infty).$$

From the reversibility $R \circ P^2 \circ R \circ P^2 = \mathrm{Id}$, a similar calculation yields

$$\frac{\partial P^2}{\partial c}(c_\infty, w) + DP^2(c_\infty, w) \cdot R \cdot \frac{\partial P^2}{\partial c}(c_\infty, v_\infty) = 0,$$

from which we have

$$\frac{\partial}{\partial c} P^4(c_\infty, v_\infty) = DP^2(c_\infty, w) \cdot (I - R) \cdot \frac{\partial P^2}{\partial c}(c_\infty, v_\infty),$$

and therefore we obtain

$$\frac{\partial P_1^4}{\partial c}(c_\infty, v_\infty) = 2\frac{\partial P_1^2}{\partial x}(c_\infty, w) \cdot \frac{\partial P_1^2}{\partial c}(c_\infty, v_\infty). \tag{43}$$

In a similar manner, using $\frac{\partial P^2}{\partial y}(c_\infty, v_\infty) = 0$ which is used in the proof of Lemma 10, we also obtain

$$\frac{\partial^2 P_1^4}{\partial y^2}(c_\infty, v_\infty) = 2\frac{\partial P_1^2}{\partial x}(c_\infty, w) \cdot \frac{\partial^2 P_1^2}{\partial y^2}(c_\infty, v_\infty). \tag{44}$$

We claim that $\frac{\partial P_1^2}{\partial x}(c_\infty, w) \neq 0$. Once this is proven, clearly the condition (42) follows from the conditions (33), (34), and from (43), (44). From Lemma 10, we have

$$DP^2(c_\infty, w) \cdot DP^2(c_\infty, v_\infty) = \begin{bmatrix} 1 & 0 \\ K & 1 \end{bmatrix}.$$

From the reversibility $R \circ P^2 \circ R \circ P^2 = \mathrm{Id}$, we also have

$$R \cdot DP^2(c_\infty, w) \cdot R \cdot DP^2(c_\infty, v_\infty) = I.$$

The claim easily follows from these and the fact that $K \neq 0$. This completes the proof. ∎

All the results obtained in this section can be summarized as:

Theorem 12 *There exist a parameter value $c = c_\infty$ and a saddle-node periodic orbit γ_* for $c = c_\infty$, such that conditions (C1) and (C2) are satisfied.*

8 The existence of a cusp-transverse heteroclinic chain

In this section we present a computer assisted proof of the existence of cusp-transverse heteroclinic chain involving the equilibrium points x_+ and x_- and the periodic orbit which is born at the bifurcation parameter value c_∞. Let us denote by $v_\infty = (0, y_\infty)$ the fixed point for the Poincaré map P^4 which is born for the parameter value c_∞.

In this section the symbol $W^u(v_\infty)$ stands for the unstable set of v_∞ in the section Θ for the map $P^4(v_\infty, \cdot)$, and $W^s(x_-)$ does the stable manifold of the equilibrium point x_- of the Michelson system (1).

The goal of this section is to prove the following theorem.

Theorem 13 *The sets $W^u(v_\infty)$ and $W^s(x_-) \cap \Theta$ have topologically transverse intersection on Θ.*

The proof of the above theorem consists of two main steps and several numerical lemmas which will be presented in the next subsections.

8.1 An estimation of $W^u(v_\infty)$ near the bifurcation point

In the first step we will construct an estimation for the unstable set $W^u(v_\infty)$ near the bifurcation point. From [5] we know that $W^u(v_\infty)$ has a topology of the half-line, with the point v_∞ at its origin and that $W^u(v_\infty) \cup W^s(v_\infty)$ is a curve with a cusp-singularity at v_∞. The proof of this fact is based on the embedding of a local diffeomorphism into the flow of a vector field. See [5, §2] and references therein for the details, and [6] for the general theory. In particular, the local reversible diffeomorphism φ around the saddle-node fixed point is C^∞-conjugate to the time-one map of the flow generated by a local vector field around a singularity. Due to the reversibility, it turns out that the corresponding vector field singularity has the double zero nilpotent linear part

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and its stable and unstable sets form a cusp, just as in the case of the Bogdanov-Takens nilpotent singularity of a planar vector field [3]. Therefore, the proof of the cusp structure of $W^u(v_\infty)$ and $W^s(v_\infty)$ is rather indirect, and it may not be so easy to give any precise estimates for $W^u(v_\infty)$. An approximate shape of $W^u(v_\infty)$ is shown in Figure 4.

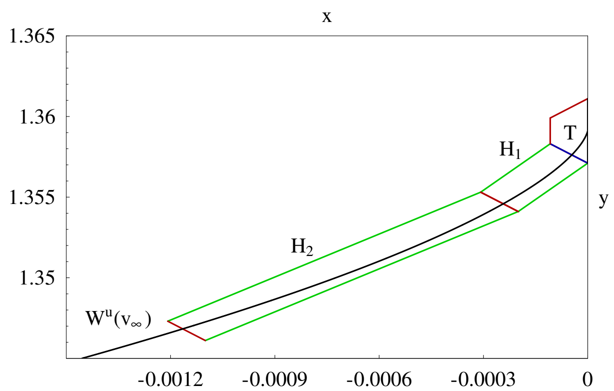

Figure 4: The sets T, H_1, H_2. The numerical evidence indicating that $W^u(v_\infty)$ forms a horizontal disk in H_2.

We assume that we have a parameterization $s : [0, \infty) \to W^u(v_\infty)$, such that s is an immersion into $\mathbb{R}^2$ and $s(0) = v_\infty$.

Definition 14 For $v_1, v_2 \in W^u(v_\infty)$, by $[v_1, v_2]$ we will denote the segment of $W^u(v_\infty)$ connecting v_1 with v_2.

Let T denotes a trapezoid (see Figures 4 and 7) with the vertices T_1, T_2, T_3, T_4 given by

$$\begin{aligned} T_1 &= (0, y_0) - u_0 + 0.2s_0 \\ T_2 &= (0, y_0) - u_0 - s_0 \\ T_3 &= (0, y_0) + u_0 + s_0 \\ T_4 &= (0, y_0) - 0.2u + s_0 \end{aligned}$$

where the vectors u_0, s_0 are

$$s_0 = (-9 \cdot 10^{-5}, 10^{-3}), \qquad u_0 = R(s_0)$$

and y_0 is the center point of the interval Y, defined in (32).

Remark 14 It should be noted that we do not know the exact location of the saddle-node point v_∞. From Lemma 9 we know that $v_\infty \in \{0\} \times Y$ and from the definition of T it follows that $v_\infty \in \{0\} \times Y \subset T$.

Next we define two h-sets $H_i = \mathfrak{h}(q_i, u_i, s_i)$, $i = 1, 2$ where

$$\begin{aligned} u_1 &= (10^{-4}, 1.5 \cdot 10^{-3}) \\ s_1 &= \frac{3}{5} \cdot (-9 \cdot 10^{-5}, 10^{-3}) = \frac{3}{5} s_0 \\ u_2 &= (4.5 \cdot 10^{-4}, 4 \cdot 10^{-3}) \\ s_2 &= s_1 \\ q_1 &= T_2 - u_1 + s_1 \\ q_2 &= T_2 - 2u_1 - u_2 + s_2 \end{aligned}$$

The h-sets H_1 and H_2 are chosen to be a neighborhood for a local unstable set $W^u(v_\infty)$ outside the trapezoid T. The location of the parallelograms $|H_1|$ and $|H_2|$ is presented in Figure 4. Notice, that the sets $|H_1|$ and $|H_2|$ have been chosen so that $T_{12} = H_1^r$ and $H_1^l = H_2^r$.

Comment: In fact H_1 will not be used as an h-set. We will explicitly use its edges, only.

The following lemma is the first step in the proof of Theorem 13.

Lemma 15 *There exist two points*

$$v_1, v_2 \in W^u(v_\infty)$$

such that $[v_1, v_2]$ *can be parameterized as a proper horizontal disk in* H_2.

The proof of Lemma 15 is given in Section 9.

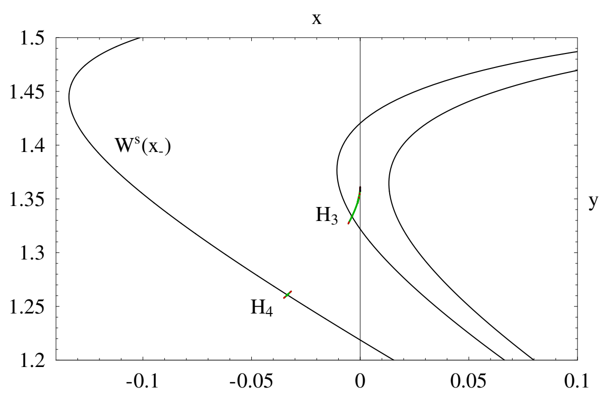

Figure 5: The location of h-sets H_3, H_4 in Θ. See also Figure 6.

8.2 Topologically transverse intersection of $W^u(v_\infty)$ and $W^s(x_-)$

Proof of Theorem 13:

By $\phi : (0,\infty) \times \mathbb{R} \times \mathbb{R}^3 \nrightarrow \mathbb{R}^3$, we will denote the local flow induced by the Michelson system (1) with the parameter value c, which is the first parameter in ϕ.

The numerical simulation shows that the stable manifold of the equilibrium point $W^s(x_-)$ intersects $P^4(c_\infty, |H_2|)$.

The idea of the proof of Theorem 13 is as follows. We choose an approximate point, say q_3 on a numerically observed intersection of $W^s(x_-)$ with $P^4(c_\infty, [v_1, v_2])$. The point q_3 will be used as the center of an h-set H_3. Then we construct a set H_4 centered at an approximate point of $P^4(c_\infty, q_3)$ and similarly H_5 will be centered at $P^2(c_\infty, q_4)$. Finally, we construct a three-dimensional h-set M centered at the equilibrium point x_- and show (a computer assisted proof) that the following sequence of covering relation holds

$$H_2 \overset{P^4(c_\infty,\cdot)}{\Longrightarrow} H_3 \overset{P^4(c_\infty,\cdot)}{\Longrightarrow} H_4 \overset{P^2(c_\infty,\cdot)}{\Longrightarrow} H_5 \overset{\Phi_H(c_\infty,\cdot)}{\Longrightarrow} M \overset{\Phi_M(c_\infty,\cdot)}{\Longrightarrow} M, \tag{45}$$

where Φ_H and Φ_M will be suitable time shifts along the flow induced by the Michelson system (1). Moreover, for all these relations, except the first one, cone conditions are satisfied.

From Lemma 15 we get that $W^u(P^4, v_\infty)$ contains a horizontal disc in H_2. Since $H_2 \overset{P^4(c_\infty,\cdot)}{\Longrightarrow} H_3$, [18, Lemma 4.7] implies that $W^u(P^4, v_\infty)$ contains a proper horizontal disk in H_3. From Theorem 8 we know that $W^s(\phi(c_\infty), x_-)$ contains a vertical disk in M satisfying the cone conditions. Therefore from Theorem 5 it follows that $W^u(v_\infty)$ and $W^s(x_-)$ have topologically transverse intersection in H_3.

Now we will precisely define the h-sets with cones listed in (45). For all these h-sets the quadratic form defining cones is $Q(x,y) = x^2 - y^2$ for $x \in \mathbb{R}^u$ and

$y \in \mathbb{R}^s$.

We define $H_i = \mathfrak{h}(q_i, s_i, u_i)$, $i = 3, 4, 5$ where

$$\begin{aligned}
q_3 &= (-0.0038, 1.3337)\\
q_4 &= (-0.0335, 1.2607),\\
q_5 &= (-0.03798, -3.10109),\\
s_3 &= s_4 = s_5 = (-10^{-4}, 2 \cdot 10^{-4}),\\
u_3 &= (10^{-4}, 4 \cdot 10^{-4}),\\
u_4 &= u_5 = (3 \cdot 10^{-4}, 5.5 \cdot 10^{-4})
\end{aligned}$$

Let $u(c)$, $s_1(c)$ and $s_2(c)$ be the eigenvectors of the linearized flow in $x_-(c) = (-c\sqrt{2}, 0, 0)$. We can find explicit formulae for these vectors, namely

$$\begin{aligned}
u(c) &= \left(\frac{3\sqrt{2}Q_1(c)}{cQ_2(c)} + \frac{Q_2(c)}{6\sqrt{2}cQ_1(c)}, \frac{6Q_1(c)}{Q_2(c)}, 1 \right)\\
s_1(c) &= \Re \left(\frac{-1}{\sqrt{2}cQ_3(c)} - \frac{Q_3(c)}{\sqrt{2}c}, \frac{-1}{Q_3(c)}, 1 \right)\\
s_2(c) &= \Im \left(\frac{-1}{\sqrt{2}cQ_3(c)} - \frac{Q_3(c)}{\sqrt{2}c}, \frac{-1}{Q_3(c)}, 1 \right)
\end{aligned}$$

where $\Re(z)$ is the real part and $\Im(z)$ is the imaginary part of the complex vector $z \in \mathbb{C}^3$ and

$$\begin{aligned}
Q_1(c) &= \left(27\sqrt{2}c + \sqrt{108 + 1458c^2} \right)^{\frac{1}{3}}\\
Q_2(c) &= -6 \cdot 2^{\frac{1}{3}} + 2^{\frac{2}{3}} Q_1(c)^2\\
Q_3(c) &= \left(1 + i\sqrt{3}\right) \left(2^{\frac{2}{3}} Q_1(c)\right)^{-1} + \left(1 - i\sqrt{3}\right) Q_1(c) \left(6 \cdot 2^{\frac{2}{3}}\right)^{-1}
\end{aligned}$$

We define a three-dimensional h-set built on these vectors with the parameter value close to c_∞, namely

$$M = \mathfrak{h}(x_-(c_0), 0.1u(c_0), 0.2s_1(c_0), 0.2s_2(c_0))$$

where c_0 is a center point of C – see (32).

We define $\Phi_H(c, x) = \phi(c, 7, x)$ and $\Phi_M(c, x) = \phi(c, 1.4, x)$. Using the algorithms presented in [20, 19] we proved (45) with a computer assistance – see also Figure 6.

∎

9 Proof of Lemma 15

The main step in the proof of Lemma 15 is to show that for $c = c_\infty$ there exists a Lyapunov function in the neighborhood of the bifurcation v_∞ for P^4. In fact, we show that the existence of a Lyapunov function is a general phenomenon for this bifurcation type. By a Lyapunov function we understand a function, which increases along a nonconstant trajectory.

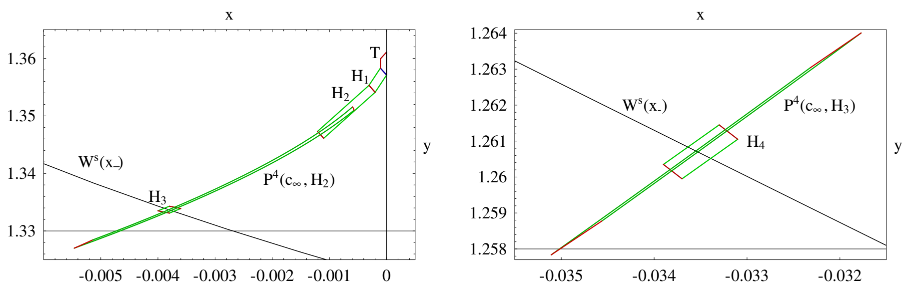

Figure 6: Covering relations $H_2 \overset{P^4(c_\infty,\cdot)}{\Longrightarrow} H_3 \overset{P^4(c_\infty,\cdot)}{\Longrightarrow} H_4$.

9.1 The existence of a Lyapunov function in the neighborhood of the bifurcation point v_∞ for $c = c_\infty$

It should be mentioned here that the existence of Lyapunov function was already established by Fontich [7], where it was obtained under certain assumptions about normal forms of the map at the degenerate fixed point (which in our context is a bifurcation point v_∞). Contrary to Fontich approach in our work we just use coordinates which bring the linearization at the bifurcation point to a Jordan form and it turns out that one of the coordinates is a desired Lyapunov function for some iterate of our map.

Lemma 16 *Let $Q : \mathbb{R}^2 -\!\!\circ\!\!\to \mathbb{R}^2$ be a C^2-map and let $(0,0)$ be a fixed point of Q, such that*

$$DQ(0,0) = \begin{bmatrix} 1 & 0 \\ K & 1 \end{bmatrix}. \tag{46}$$

If $\frac{\partial^2 Q_1}{\partial y^2}(0,0) > 0$ and $K \neq 0$, then there exists a neighborhood U of $(0,0)$ and an integer $n > 0$ such that for $v = (v_1, v_2) \in U$, $v \neq 0$ holds $Q_1^n(v) > v_1$.

Proof: Let $B(r,r) = D^2Q(0,0)(r,r)$ be the vector valued bilinear form induced by the second order derivative of Q at $(0,0)$ and $A = DQ(0,0)$. We have

$$Q(r) = Ar + \frac{1}{2}B(r,r) + o(|r|^2)$$

An easy computation shows that

$$\begin{aligned} Q^n(r) = A^n r + \frac{1}{2}A^{n-1}B(r,r) + \frac{1}{2}A^{n-2}B(Ar,Ar) + \cdots + \\ + \frac{1}{2}AB(A^{n-2}r, A^{n-2}r) + \frac{1}{2}B(A^{n-1}r, A^{n-1}r) + o(|r|^2) \end{aligned} \tag{47}$$

From (46) we have

$$A^n = \begin{bmatrix} 1 & 0 \\ nK & 1 \end{bmatrix}. \tag{48}$$

Consider now, Q_1, the first coordinate of Q. From above formulas we obtain

$$Q_1^n(r) = r_1 + \frac{1}{2}r^T S(n)r + o(|r|^2), \tag{49}$$

where $r = (r_1, r_2)^T$ and $S(n)$ a symmetric 2×2 matrix given by

$$S(n) = B_1 + A^T B_1 A + \cdots + \left(A^{n-1}\right)^T B_1 A^{n-1}. \tag{50}$$

From now on we will drop the index in B_1. This means that $B_{ij} = \frac{\partial^2 Q_1}{\partial x_i \partial x_j}$, for $i, j = 1, 2$ and $x_1 = x$, $x_2 = y$.

We would like to show now that there exists n, such that $S(n)$ is positively definite, i.e. $S(n)(r, r) > 0$ for $r \neq 0$. Let

$$B = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

We have

$$(A^i)^T B A^i = \begin{bmatrix} ci^2K^2 + 2biK + a & iKc + b \\ iKc + b & c \end{bmatrix}.$$

From the above equation we obtain

$$S(n) = \begin{bmatrix} \frac{K^2cn^3}{3} + c_1n^2 + c_2n + c_3 & \frac{Kcn^2}{2} + c_4n + c_5 \\ \frac{Kcn^2}{2} + c_4n + c_5 & nc \end{bmatrix}$$

for suitable constants $c_1, \ldots, c_5$. The determinant of $S(n)$ has the following form

$$\det S(n) = \frac{K^2c^2n^4}{12} + d_3n^3 + d_2n^2 + d_1n + d_0$$

for some constants $d_0, \ldots, d_3$. This shows that for a sufficiently large n $\det S(n)$ is positive. Recall that $c = \frac{\partial^2 Q_1}{\partial y^2}(0, 0) > 0$. Hence, $S(n)_{11} > 0$ for sufficiently large n. This proves that $S(n)$ is positively definite for n large enough.

Till now we have shown that $D^2Q_1^n(0, 0)$ is positively definite. Let U be a convex neighborhood containing $(0, 0)$ (in fact star-shaped with respect to $(0, 0)$ is enough) such that any quadratic symmetric matrix G satisfying

$$G \in \left[D^2Q_1^n(U)\right]_I \tag{51}$$

is positively definite.

From the Taylor formula and (48) it follows that for $(x, y) \in U \setminus \{(0, 0)\}$ there exits a matrix G satisfying (51) such that

$$Q_1^n(x, y) - x = \frac{1}{2}G((x, y), (x, y)) > 0$$

∎

Remark 17 If in the above lemma the sign in the assumption $\frac{\partial^2 Q_1}{\partial y^2}(0,0) > 0$ is reversed, i.e. we assume that $\frac{\partial^2 Q_1}{\partial y^2}(0,0) < 0$, then we obtain $Q_1^n(v) < v_1$ for $v \in U$, $v \neq 0$.

In the context of cocoon bifurcation for (1) Lemma 16 and Remark 17 suggest that the first coordinate should be a Lyapunov function for P^{4n}, for some $n > 0$, in a small neighborhood U of v_∞. It turns that this can be proven for $n = 1$ and the neighborhood T of the bifurcation point. On the other hand the sets H_1 and H_2 are separated from the bifurcation point v_∞, hence one can try to prove that the first coordinate is a Lyapunov function on H_1 and H_2 by a direct verification of either $P_1^4(x,y) < x_1$ or $P_1^4(x,y) > x_1$ using interval arithmetics, because if valid one of these inequalities is satisfied with some nonzero margin. The following lemma summarizes these observations and it is the main step in the proof of Lemma 15.

Lemma 18 *For $v = (v_1, v_2) \in T \cup H_1 \cup H_2$, $v \neq v_\infty$, it holds that*

$$P_1^4(c_\infty, v) < v_1.$$

Proof: First we focus on T. Here, since $v_\infty \in T$ is a fixed point, we cannot hope that $P^4(v_1, v_2) - v_1 < -\delta$ for some $\delta > 0$. Therefore we use the second derivative as in the final part of the proof of Lemma 16. For this end we cover T with two star-shaped sets with respect to v_∞ and on each of them we verify that interval enclosure of the second derivative of the proposed Lyapunov function is negatively definite.

Let us denote by T^t and T^b the subsets of the trapezoid T given by

$$\begin{aligned} T^t &= \{(x,y) \in T \mid y \geq \min Y\} \\ T^b &= \{(x,y) \in T \mid y \leq \max Y\}, \end{aligned}$$

where Y is defined in (32) as the rigorous bound for the saddle-node point $(0, v_\infty)$ with $v_\infty \in Y$.

Obviously $T \subset T^t \cup T^b$. With a computer assistance we proved that for all $c \in C$ we have the following estimations

$$\left[D^2 P_1^4(c,\cdot)(T^t)\right] \subset \begin{bmatrix} \mathbf{a_1} & \mathbf{b_1} \\ \mathbf{c_1} & \mathbf{d_1} \end{bmatrix}, \qquad \left[D^2 P_1^4(c,\cdot)(T^b)\right] \subset \begin{bmatrix} \mathbf{a_2} & \mathbf{b_2} \\ \mathbf{c_2} & \mathbf{d_2} \end{bmatrix}$$

where

$$\begin{aligned} \mathbf{a_1} &= [-780.16131381343655, -709.93527387806591] \\ \mathbf{b_1} &= [-89.947975019107204, -66.727549530917358] \\ \mathbf{c_1} &= [-89.947849107194429, -66.727364301250489] \\ \mathbf{d_1} &= [-19.331688841150434, -11.453819205730355] \\ \mathbf{a_2} &= [-778.05323373414433, -714.78668326675711] \\ \mathbf{b_2} &= [-101.63969051099849, -79.522565868200005] \\ \mathbf{c_2} &= [-101.63977655838916, -79.522402280329359] \\ \mathbf{d_2} &= [-22.635893897013542, -15.224239418346446] \end{aligned} \tag{52}$$

An easy computation show that each symmetric matrix $G \in \left[D^2 P_1(c,\cdot)(T^t)\right]$ or $G \in \left[D^2 P_1(c,\cdot)(T^b)\right]$ is negatively definite. Moreover, from Lemma 10 we have $K = \frac{\partial P_2^4}{\partial x}(c_\infty, v_\infty) \neq 0$. From Lemma 16 we obtain that for all $v = (v_1, v_2) \in T$ and $v \neq v_\infty$

$$P_1^4(c_\infty, v) < v_1$$

To finish the proof we check in rigorous computation that $P_1^4(c, v) < v_1$ for all $c \in C$ and $v \in H_1 \cup H_2$.

Since H_1 and H_2 are separated from the fixed point we were able to verify condition $P_1(v_1, v_2) - v_1 < -\delta$ in direct $\mathcal{C}^0$ computations. We covered the set H_1 by 67290 nonequal pieces (smaller when closer to fixed point) and for each element in such a grid we verified inequality

$$P_1^4([v_1],[v_2]) - [v_1] < -1.3233105116707521 \cdot 10^{-7},$$

where $([v_1],[v_2])$ is an element of a grid.

The set H_2 has been covered by 5000 equal parts and for each element in such a grid we verified inequality

$$P_1^4([v_1],[v_2]) - [v_1] < -1.1552589508338426 \cdot 10^{-5}.$$

∎

Remark 19 Notice that the proof of the existence of the estimations (52) is the most time-consuming part of the numerical proof. In fact, we divided both the sets T^t and T^b into 1913 equal pieces as well as their images $P^2(C, T^t)$ and $P^2(C, T^b)$ into 990 and 832 parts, respectively. Next we compute the hessian $D^2 P_1^4(C, T)$ by composition of two Poincaré maps P^2 and its partial derivatives on a suitable sets.

9.2 The proof of Lemma 15

Recall that by $s : [0, \infty) \to W^u(v_\infty)$ we denoted a parameterization of the unstable set of v_∞ for $P^4(c_\infty, \cdot)$ on the plane Θ.

Definition 15 Let $Z \subset \mathbb{R}^2$ be closed and $Y \subset \partial Z$. Assume that $s(t_0) \in W^u(v_\infty) \cap Z$ for some $t_0 \geq 0$.

We say that $W^u(v_\infty)$ *leaves* Z *through* Y, if the following conditions are satisfied:

$$t_e = \sup\{t : t \geq t_0, s([t_0, t]) \subset Z\} < \infty$$
$$s(t_e) \in Y.$$

The point $s(t_e)$ will be called *the exit point.* If $Y = \partial Z$, then we just say that $W^u(v_\infty)$ leaves Z.

Let us introduce the following notation. By $T_{ij}(v)$ we will denote the edge of a trapezoid $T(v)$ connecting $T_i(v)$ with $T_j(v)$.

Proof of Lemma 15: Let us denote by N the set

$$N = T \cup H_1 \cup H_2$$

The proof of the lemma consists of the following steps:

1. we will show that the maximal invariant set for $P^4(c_\infty, \cdot)$ in N is a single point v_∞. Hence, $W^u(v_\infty)$ must leave the set N.
2. we will show that $W^u(v_\infty)$ must leave the set N through the left edge H_2^l.
3. we conclude that this implies that some part of $W^u(v_\infty)$ is a horizontal disk in H_2.

From Lemma 18 it follows that

$$P_1^4(c_\infty, z) < z_1 \quad \text{for } (z_1, z_2) \in N \setminus \{v_\infty\}.$$

This shows by a standard Lyapunov function argument that $W^u(v_\infty)$ must leave the set N, hence there exists a point $v_2 \in \partial N \cap W^u(v_\infty)$ such that

$$[v_\infty, v_2] \subset N. \tag{53}$$

Next we will show that $v_2 \in H_2^l$. Using the algorithms presented in [20, 19] we prove with a computer assistance that for all $c \in C$ the following conditions hold (see Figures 7 and 8):

$$\begin{aligned} P^{-4}(c, H_1^b \cup H_2^b \cup H_2^t) \cap N = \emptyset, \\ P^{-8}(c, H_1^t \cup T_{14} \cup T_{34}) \cap N = \emptyset. \end{aligned} \tag{54}$$

Since for $z \in N \setminus \{v_\infty\}$, $P_1^4(c_\infty, z) < 0$ we get $v_2 \notin T_{23}$. From (53) we obtain

$$P^{-4n}([v_\infty, v_2]) \subset [v_\infty, v_2] \subset N$$

for $n > 0$. From this and (54), it follows that

$$v_2 \notin H_1^t \cup H_1^b \cup H_2^t \cup H_2^b \cup T_{14} \cup T_{34},$$

and therefore $v_2 \in H_2^l$.

It remains to show that there exists $v_1 \in W^u(v_\infty)$ such that $[v_1, v_2]$ can be parameterized as a proper horizontal disk in H_2. Since v_∞ and v_2 lie in two disjoint components of $N \setminus H_2^r$ and since $[v_\infty, v_2] \subset N$, it follows that there exists at least one point

$$\tilde{v} \in W^u(v_\infty) \cap H_2^r, \tag{55}$$

hence

$$D = \bigcap \{[w, v_2] : w \in W^u(v_\infty) \cap H_2^r\}$$

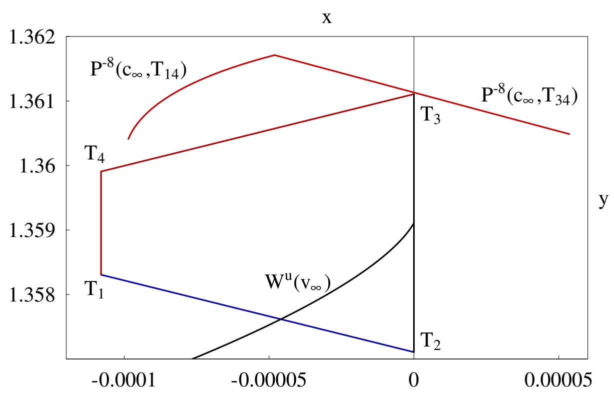

Figure 7: The trapezoid T, unstable set $W^u(v_\infty)$ and the preimage of the two edges of T.

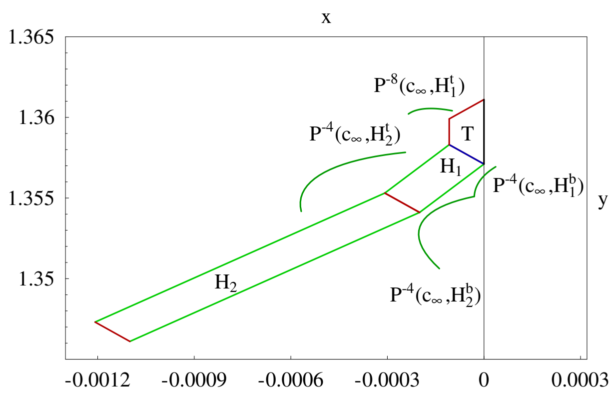

Figure 8: The sets H_1, H_2 and the preimages of their horizontal edges $H_i^t \cup H_i^b$, $i = 1, 2$.

is a nonempty set. Moreover, there exists $v_1 \in W^u(v_\infty) \cap H_2^r$ such that $D = [v_1, v_2]$. We will show that $[v_1, v_2] \subset H_2$. Assume it is not the case, i.e. there exists $v_3 \in [v_1, v_2]$ such that $v_3 \in N \setminus H_2$. Hence, v_3 and v_2 lie in two disjoint components of $N \setminus H_2^r$. Then, there exists $v_4 \in W^u(v_\infty) \cap H_2^r$ such that $[v_4, v_2] \subsetneq [v_3, v_2] \subsetneq [v_1, v_2]$. This contradicts the choice of v_1.

We have shown that $[v_1, v_2] \subset H_2$, $v_1 \in H_2^r$ and $v_2 \in H_2^l$. Since $[v_1, v_2]$ is a connected part of the one-dimensional manifold, it follows that there exists a continuous function $f : [-1, 1] \to H_2$ such that $f([-1, 1]) = [v_1, v_2]$. This completes the proof. Observe also that $[v_1, v_2]$ has an empty intersection with H_2^t and H_2^b, hence it is a proper disk. ∎

9.3 Technical data.

In order to compute Poincaré maps P and $\tilde{\Pi}$ with their partial derivatives and time translations Φ_M, Φ_H, we used the interval arithmetic [9, 14], set algebra and the Lohner algorithms [11, 22] developed at the Jagiellonian University by the CAPD group [2]. The C++ source files of the program with an instruction how it should be compiled and run are available at [17]. All computations were performed with the Pentium IV, 3GHz processor and 512MB RAM under Mandriva Linux 2006 with gcc-4.0.1 and MS Windows XP Professional with gcc-3.4.4. The computations took approximately 65 minutes.

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