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# Attracting Fixed Points for the Kuramoto–Sivashinsky Equation: A Computer Assisted Proof*

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**Abstract.** We present a computer assisted proof of the existence of several attracting fixed points for the Kuramoto–Sivashinsky equation

$$u_t = (u^2)_x - u_{xx} - \nu u_{xxxx}, \quad u(x,t) = u(x+2\pi,t), \quad u(x,t) = -u(-x,t),$$

where $\nu > 0$. The method is general and can be applied to other dissipative PDEs.

**Key words.** dissipative PDEs, fixed points, Galerkin projection, computer assisted proof

**AMS subject classifications.** 35B35, 35B45, 65G20, 65N30

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## 1. Introduction.

The goal of this paper is to extend the method of self-consistent a priori bounds developed in [ZM, Z] for a rigorous study of dynamics of dissipative PDEs. We present an approach which allows us to show that a given fixed point for a PDE is asymptotically stable. We apply the method to the Kuramoto–Sivashinsky (KS) equation subject to periodic and odd boundary conditions

$$u_t = (u^2)_x - u_{xx} - \nu u_{xxxx}, \quad u(x,t) = u(x+2\pi,t), \quad u(x,t) = -u(-x,t), \tag{1.1}$$

where $\nu > 0$. While the method will be explained in detail later, we would like to stress here its basic ingredients, which will also explain how this paper is dependent on [Z] and [ZM] and what is new here.

The approach starts as in [ZM]:

1. We have to find an approximate attracting fixed point $x_0$ for some Galerkin projection of (1.1). Then we construct a trapping region around $x_0$ (which is an example of self-consistent a priori bounds defined in [ZM]) using the algorithm presented in [ZM]. From this we conclude that there exists a fixed point $x^* \in R$, but we cannot claim its asymptotic stability.
2. In paper [Z], we obtained estimates for the Lipschitz constants for the flow induced by the Navier–Stokes equations on two-dimensional torus. Here we adopt this approach to construct a norm for which the induced flow is a contraction around $x^*$.

In the present work, for each attracting branch from a nonrigorous steady state bifurcation diagram presented in [JKT], we picked up a point on it, and we proved that it is attracting.

Below we include some of the attracting steady states we had proved rigorously to exist.

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- $\nu \in 0.75 + [-10^{-2}, 10^{-2}]$, two stable unimodal fixed points.
- $\nu \in 0.5 + [-10^{-4}, 10^{-4}]$, two stable unimodal fixed points.
- $\nu \in 0.3 + [-10^{-4}, 10^{-4}]$, two stable unimodal fixed points.
- $\nu \in 0.125 + [-10^{-4}, 10^{-4}]$, one stable bimodal fixed point.
- $\nu \in 0.1 + [-10^{-4}, 10^{-4}]$, one bimodal stable fixed point.
- $\nu \in 0.08 + [-10^{-6}, 10^{-6}]$, one bimodal stable fixed point. A pair of stable fixed points close to $R_3 t_2$ (see [JKT]).
- $\nu \in 0.062 + [-10^{-6}, 10^{-6}]$, two stable trimodal points and two stable points from giant branch.
- $\nu \in 0.045 + [-10^{-6}, 10^{-6}]$, $\nu \in 0.04 + [-10^{-7}, 10^{-7}]$, two stable points from giant branch.

In the above listing, when we write that, for $\nu \in 0.75 + [-10^{-2}, 10^{-2}]$, we have two stable fixed points, this means that, for all $\nu$ in this interval, these stable fixed points exist. In section 4, we present an example of a precise theorem about an existence of a fixed point obtained using our method.

In sections 2 and 3, we present the method in detail, and we prove Theorem 3.8, which is the main tool in our approach. In section 4, we present an example of a precise theorem, give an outline of the algorithm, and present numerical data from the proof. In section 5, we derive various estimates for the KS equation required in the rigorous check of assumptions of Theorem 3.8. In section 6, we discuss the directions in which this work can be extended further.

**2. Uniform convergence of Galerkin projections on a trapping region.** We adopt here the notation used in sections 4 and 5 in [Z]. Let $H$ be a real Hilbert space. Let $e_1, e_2, \dots$ form an orthonormal basis in $H$.

In what follows, we will quite often denote the elements of $H$ by $x$, and we hope it will not be confused with the space variable in (1.1).

Let $A_n : H \to \mathbb{R}$ denote a projection onto a one-dimensional subspace $\langle e_n \rangle$; i.e., $x = \sum A_n(x) e_n$ for all $x \in H$. By $X_n$ we will denote a space spanned by $\{e_1, \dots, e_n\}$. Let $P_n$ denote the projection onto $X_n$, $Q_n = I - P_n$.

For $x \in \mathbb{R}^n$ or $x \in H$, we set $|x|$ to be a standard (Euclidean) norm, $|x|_\infty = \max_i |x_i|$ and $|x|_1 = \sum_i |x_i|$.

We investigate the Galerkin projections of the problem

$$x' = F(x) = L(x) + N(x), \tag{2.1}$$

where $L$ is a linear operator and $N$ is a nonlinear part of F. We assume that the basis $e_1, e_2, \dots$ of $H$ is built from eigenvectors of $L$. We assume that the corresponding eigenvalues $\lambda_k$ (i.e., $Le_k = \lambda_k e_k$) are ordered so that

$$\lambda_1 \geq \lambda_2 \geq \dots \quad \text{and} \quad \lim_{k \to \infty} \lambda_k = -\infty.$$

Hence $L$ can have only a finite number of positive eigenvalues.

Definition 2.1. *Let $W \subset H$ and $F : dom(F) \to H$, $W$ be closed. We say that $W$ and $F$ satisfy conditions* C1, C2, *and* C3 *if the following hold:*

C1. *There exists* $M \geq 0$ *such that* $P_n(W) \subset W$ *for* $n \geq M$.

C2. *Let* $\hat{u_k} = \max_{x\in W} |A_k x|$. *Then* $\hat{u} = \sum \hat{u_k} e_k \in H$. *In particular,* $|\hat{u}| < \infty$.

C3. *The function* $x \mapsto F(x)$ *is continuous on* $W$, *and* $f = \sum_k f_k e_k$, *given by* $f_k = \max_{x\in W} |A_k F(x)|$, *is in* $H$. *In particular,* $|f| < \infty$.

Observe that, if $W_m \subset X_m$ and $\{a_k^-, a_k^+\}$ form self-consistent a priori bounds (see [ZM, Def. 2.1]) for $F$, then $W = W_m \oplus \Pi_{k=m+1}^{\infty}[a_k^-, a_k^+]$ and $F$ satisfy conditions C1, C2, and C3.

Definition 2.2. *We say that* $W \subset H$ *and* $F = N + L$ *satisfy condition* D *if, for any* $i, j$, *the function*

$$\frac{\partial N_i}{\partial x_j} : W \to \mathbb{R} \tag{2.2}$$

*is continuous and the following condition holds:*

D. *There exists* $l \in \mathbb{R}$ *such that, for all* $k = 1, 2, \ldots,$

$$1/2 \sum_{i=1}^{\infty} \left| \frac{\partial N_k}{\partial x_i} \right| (W) + 1/2 \sum_{i=1}^{\infty} \left| \frac{\partial N_i}{\partial x_k} \right| (W) + \lambda_k \leq l. \tag{2.3}$$

The main idea behind condition D is to ensure that Lipschitz constants of flows induced by Galerkin projections of (2.1) are uniformly bounded. (See the proof of Theorem 13 in [Z] for more details.)

Definition 2.3. *Consider an ODE*

$$x' = f(x), \tag{2.4}$$

*where* $x \in \mathbb{R}^n$. *The compact set* $W \subset \mathbb{R}^n$ *is called* a trapping region for (2.4) *if, for any solution* $x(t)$ *of* (2.4), *if* $x(0) \in W$, *then* $x(t) \in W$ *for all* $t > 0$.

The following easy lemma was used throughout this paper as a criterion for a set to be a trapping region.

Lemma 2.4. *Assume that* $W$ *is a closure of an open set, with a piecewise smooth boundary. For any* $x \in \partial W$, *let* $\nu(x)$ *denote an outward normal vector to* $\partial W$.

*If, for all* $x \in \partial W$, *we have* $\nu(x) \cdot f(x) < 0$, *then* $W$ *is a trapping region for* (2.4).

The following theorem was proved in [Z].

Theorem 2.5 (see [Z, Theorem 13]). *Assume that* $R \subset H$ *and* $F$ *satisfy conditions* C1, C2, C3, *and* D *and that* $R$ *is convex. Assume that* $P_n(R)$ *is a trapping region for the* $n$*-dimensional Galerkin projection of* (2.1) *for all* $n > M_1$. *Then the following hold.*

1. Uniform convergence and existence. *For a fixed* $x_0 \in R$, *let* $x_n : [0, \infty] \to P_n(R)$ *be a solution of* $x' = P_n(F(x))$, $x(0) = P_n x_0$. *Then* $x_n$ *converges uniformly on compact intervals to a function* $x^* : [0, \infty] \to R$, *which is a solution of* (2.1), *and* $x^*(0) = x_0$. *The convergence of* $x_n$ *on compact time intervals is uniform with respect to* $x_0 \in R$.
2. Uniqueness within $R$. *There exists only one solution of the initial value problem* (2.1), $x(0) = x_0$ *for any* $x_0 \in R$, *such that* $x(t) \in R$ *for* $t > 0$.
3. Lipschitz constant. *Let* $x : [0, \infty] \to R$ *and* $y : [0, \infty] \to R$ *be solutions of* (2.1); *then*

$$|y(t) - x(t)| \leq e^{lt} |x(0) - y(0)|.$$

4. Semidynamical system. *The map* $\varphi : \mathbb{R}_+ \times R \to R$, *where* $\varphi(\cdot, x_0)$ *is a unique solution of* (2.1), *such that* $\varphi(0, x_0) = x_0$, *defines a semidynamical system on* $R$; *namely,*
   - $\varphi$ *is continuous,*
   - $\varphi(0, x) = x$, *and*
   - $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$.

In the context of this paper, the statement about the Lipschitz constant in Theorem 2.5 is of special importance. We can formulate it as

$$|\varphi(t, x) - \varphi(t, y)| \leq e^{lt}|x - y|, \quad t \geq 0, \tag{2.5}$$

where $l$ is given in condition D and $x, y \in R$.

Assume that we have a trapping region, $R$, satisfying the assumptions of Theorem 2.5. The next step is to prove that the induced semiflow is contracting. This may be hard to achieve in the original norm, but, in section 3, we construct another norm (similar to the $|\cdot|_\infty$-norm) for which we are able to show that (2.5) holds with $l < 0$ for the steady states for the KS equation mentioned in the introduction.

**3. Diagonalization and construction of a "contracting" norm.** As was mentioned in section 2, we would like to construct a "contracting" norm on trapping region $R$ ( $l < 0$ in (2.5)).

**3.1. Block decomposition.** Our construction will be based on the approximate diagonalization of the matrix $\frac{\partial F}{\partial x}$. We want this matrix to be dominated by diagonal terms. This is achieved by an approximate diagonalization in case of real eigenvalues. The case of the complex eigenvalues forces us to consider blocks on the diagonal. We formalize this as follows.

Definition 3.1. *A decomposition of* $H$ *into a sum of subspaces is called* a block decomposition of $H$ *if the following conditions are satisfied.*

1. $H = \bigoplus_i H_i$.
2. *For every* $i$, $h_i = \dim H_i \leq h_{\max} < \infty$.
3. *For every* $i$, $H_i = \langle e_{i_1}, e_{i_2}, \dots, e_{i_{h_i}} \rangle$.
4. *If* $\dim H = \infty$, *then there exists* $k$ *such that, for* $i > k$, $h_i = 1$.

For a block decomposition of $H$, we adopt the following notation, which makes a distinction between blocks and one-dimensional subspaces spanned by $\langle e_i \rangle$. For the blocks, we use $H_{(i)} = \langle e_{i_1}, \dots e_{i_k} \rangle$, where $(i) = (i_1, \dots i_k)$. The symbol $H_i$ will always mean the subspace generated by $e_i$. For one-dimensional block $(i)$, we adopt the following convention: the only element of $(i)$ will be denoted by the same letter $i$.

For a given block decomposition of $H$ and block $(i)$, we set

$$\dim\ (i) = \dim\ H_{(i)}.$$

For any $x \in H$, by $x_{(i)}$ we will denote a projection of $x$ onto $H_{(i)}$. For any $a$ and $(i) = (i_1, \dots, i_k)$, we will say that $(i) \leq a$ if $i_s \leq a$ for all $s = 1, \dots, k$, and we say that $(i) > a$ if $i_s > a$ for all $s = 1, \dots, k$.

On each component $H_{(i)}$, we will use a norm induced from $H$. By $P_{(i)}$ we will denote an orthogonal projection onto $H_{(i)}$. By $\mathrm{Lin}(H_{(i)}, H_{(j)})$ we denote a set of all linear maps from $H_{(i)}$ to $H_{(j)}$ equipped with an operator norm $|A| = \max_{|v|=1, v \in H_{(i)}} |Av|$.

We have the following easy lemma.

Lemma 3.2. *Assume that we have a block decomposition of $H$, and let $W \subset H$. If, for any $k, l$, the function*

$$\frac{\partial F_k}{\partial x_l} : W \to \mathbb{R}$$

*is continuous, then, for every $(i)$ and $(j)$, the map*

$$\frac{\partial F_{(i)}}{\partial x_{(j)}} : W \to Lin(H_{(j)}, H_{(i)}) \approx \mathbb{R}^{\dim\ (j) \times \dim\ (i)}$$

*is continuous.*

For any square matrix $Q \in \mathbb{R}^{\dim H \times \dim H}$ (a linear map $Q : \mathrm{dom}(Q) \to H$) and for any blocks $(i), (j)$, we define a matrix $Q_{(i)(j)}$ as the matrix corresponding to an induced linear map $Q_{(i)(j)} : H_{(j)} \to H_{(i)}$ given by $Q_{(i)(j)}(x) = P_{(i)}(QP_{(j)}x)$.

**3.2. A block-infinity norm for block decomposition.** For a fixed block decomposition of $H$, we define the norm (*the block-infinity norm*) by

$$|x|_{b,\infty} = \max_{(i)} |P_{(i)}x|. \tag{3.1}$$

We have the following easy lemma.

Lemma 3.3. *Assume that $W \subset H$, $W$ is closed and satisfies condition* C2. *Then on $W$ the convergence in the norm $|\cdot|$ is equivalent to the convergence in the norm $|\cdot|_{b,\infty}$; namely, for any sequence $\{x_n\} \subset W$, $|x_n - x^*| \to 0$ if and only if $|x_n - x^*|_{b,\infty} \to 0$.*

Now we turn to the computation of the logarithmic norm for $|\cdot|_{b,\infty}$.

For any norm $\|\cdot\|$ on $\mathbb{R}^n$ following [HNW], we introduce the notion of the logarithmic norm of a matrix by the following definition.

Definition 3.4. *Let $Q$ be a square matrix; then we call*

$$\mu(Q) = \limsup_{h>0, h\to 0} \frac{\|I + hQ\| - 1}{h} \tag{3.2}$$

*the* logarithmic norm *of $Q$.* Definition 3.4 differs slightly from Definition I.10.4 in [HNW] because, to avoid a question of the existence of the limit in (3.2), we use the lim sup.

By $\mu(Q)$ we denote the logarithmic norm induced by the Euclidean norm, and for all other norms we will use a subscript identifying it.

The following theorem was proved in [HNW].

Theorem 3.5 (see [HNW, Th. I.10.5]). *The logarithmic norm is obtained by the following formulas:*

$$\mu(Q) = \textit{the largest eigenvalue of} \quad 1/2(Q + Q^T), \tag{3.3}$$

$$\mu_\infty(Q) = \max_k \left( q_{kk} + \sum_{i, i\neq k} |q_{ki}| \right), \tag{3.4}$$

$$\mu_1(Q) = \max_i \left( q_{ii} + \sum_{k, k\neq i} |q_{ki}| \right). \tag{3.5}$$

The next lemma tells us how to compute the logarithmic norm induced by the block-infinity norm.

Lemma 3.6. *Assume that we have a block decomposition of* $\mathbb{R}^n$*; then*

$$\mu_{b,\infty}(Q) \leq \max_{(i)} \left( \mu(Q_{(i)(i)}) + \sum_{(k)\neq(i)} |Q_{(i)(k)}| \right). \tag{3.6}$$

*Proof.* Since, for any $h > 0$, $(i)$, and $x \in \mathbb{R}^n$, we have

$$\begin{aligned} |P_{(i)}(I + hQ)x| &= \left| \left( I_{(i)(i)} + hQ_{(i)(i)} \right) x_{(i)} + h \sum_{(j),(j)\neq(i)} Q_{(i)(j)} x_{(j)} \right| \\ &\leq \left| \left( I_{(i)(i)} + hQ_{(i)(i)} \right) x_{(i)} \right| + h \sum_{(j),(j)\neq(i)} \left| Q_{(i)(j)} x_{(j)} \right| \\ &\leq \left( \left| I_{(i)(i)} + hQ_{(i)(i)} \right| + h \sum_{(j),(j)\neq(i)} \left| Q_{(i)(j)} \right| \right) |x|_{b,\infty}, \end{aligned}$$

then

$$|(I + hQ)|_{b,\infty} \leq \max_{(i)} \left( \left| I_{(i)(i)} + hQ_{(i)(i)} \right| + h \sum_{(j),(j)\neq(i)} \left| Q_{(i)(j)} \right| \right). \tag{3.7}$$

From the above equation and the definition of the logarithmic norm, one easily obtains the assertion of the theorem. ■

From Theorem 3.5, it follows that, when all blocks are one-dimensional, we have an equality in (3.6), but observe that this is not true in general as is shown by the following example.

Let $n = 4$. Consider the blocks $(e_1, e_2)$, $(e_3)$, and $(e_4)$ and the matrix

$$Q = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.8}$$

An easy computation shows that

$$\mu_{b,\infty}(Q) = 2 < \max_{(i)} \left( \mu(Q_{(i)(i)}) + \sum_{(k)\neq(i)} |Q_{(i)(k)}| \right) = 2\sqrt{2}.$$

**3.3. Lipschitz constants in block-infinity norm and main theorem .** The following theorem has exactly the same proof as Theorem 13 in [Z]. The only difference is that the standard norm in $H$ is replaced by the block-infinity norm.

Theorem 3.7. *Assume that* $R \subset H$*,* $R$ *is convex, and* $F$ *satisfies conditions* C1, C2, *and* C3. *Assume that we have a block decomposition of* $H$ *such that condition* Db *holds.*

Db. *There exists $l \in \mathbb{R}$ such that, for any $(i)$ and $x \in R$,*

$$\mu\left(\frac{\partial F_{(i)}}{\partial x_{(i)}}(x)\right) + \sum_{(k),\,(k)\neq(i)} \left|\frac{\partial F_{(i)}}{\partial x_{(k)}}(x)\right| \leq l.$$

*Assume that $P_n(R)$ is a trapping region for the $n$-dimensional Galerkin projection of (2.1) for all $n > M_1$. Then the following hold.*

1. Uniform convergence and existence. *For a fixed $x_0 \in R$, let $x_n : [0,\infty] \to P_n(R)$ be a solution of $x' = P_n(F(x))$, $x(0) = P_n x_0$. Then $x_n$ converges uniformly in a max-infinity norm on compact intervals to a function $x^* : [0,\infty] \to R$, which is a solution of (2.1) and $x^*(0) = x_0$. The convergence of $x_n$ on compact time intervals is uniform with respect to $x_0 \in R$.*
2. Uniqueness within $R$. *There exists only one solution of the initial value problem (2.1), $x(0) = x_0$ for any $x_0 \in R$, such that $x(t) \in R$ for $t > 0$.*
3. Lipschitz constant. *Let $x : [0,\infty] \to R$ and $y : [0,\infty] \to R$ be solutions of (2.1); then*

$$|y(t) - x(t)|_{b,\infty} \leq e^{lt}|x(0) - y(0)|_{b,\infty}.$$

4. Semidynamical system. *The map $\varphi : \mathbb{R}_+ \times R \to R$, where $\varphi(\cdot, x_0)$ is a unique solution of (2.1), such that $\varphi(0, x_0) = x_0$, defines a semidynamical system on $R$; namely,*
   - *$\varphi$ is continuous,*
   - *$\varphi(0, x) = x$,*
   - *$\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$.*

The following theorem is the main tool in proving an existence of attracting fixed points.

Theorem 3.8. *We use the same assumptions on $R$,$F$ and a block decomposition of $H$ as in Theorem 3.7. Assume that $l < 0$.*

*Then there exists a fixed point for (2.1), $x^* \in R$, unique in $R$, such that, for every $y \in R$,*

$$|\varphi(t, y) - x^*|_{b,\infty} \leq e^{lt}|y - x^*|_{b,\infty} \quad \text{for } t \geq 0,$$
$$\lim_{t\to\infty} \varphi(t, y) = x^*.$$

*Proof.* It is enough to prove the existence of $x^* \in R$ such that $F(x^*) = 0$ because the uniqueness in $R$ and all other assertions follow directly from the assumption $l < 0$ and the Lipschitz constant estimates given in Theorem 3.7.

It is easy to see that, for all $n > M_1$, there exists $x_n \in P_n R$, a fixed point for the $n$th Galerkin projection. Passing to the limit with $n$ (by picking a subsequence eventually), we obtain $x^*$ (see [ZM, Thm 2.16] for details). ■

**4. An example of a theorem and a description of an algorithm.** In this section, we present an example of a theorem we prove using our method, give a description of an algorithm, and provide some numerical data from the proof.

Theorem 4.1. *Let*

$$u(x) = -2\left(0.711691\sin(x) - 0.123059\sin(2x) + 0.01011\sin(3x)\right).$$

*For any $\nu \in 0.75 + [-10^{-2}, 10^{-2}]$, there exists an equilibrium solution $u_\nu(x)$ to (1.1) such that*

- $u_\nu$ *is attracting,*
- $\|u_\nu - u\|_{L^2} \le 0.104$, $\|\partial_x u_\nu - \partial_x u\|_{L^2} \le 0.132$, *and* $\|u_\nu - u\|_{C^0} \le 0.084$.

The attracting fixed point from the above theorem had already been discovered (nonrigorously) by Jolly, Kevrekidis, and Titi in [JKT]. In the terminology used there, this is a unimodal fixed point.

The proof consists of two parts.

1. The first part is a construction of topologically self-consistent a priori bounds (see Definition 2.11 in [ZM]) for the KS equation, i.e., $W \subset X_m$ and $\{a_k^-, a_k^+\}_{k>m}$, an isolating block $N \subset W$ with empty exit set. We define a set $R$ given by

$$R = N \oplus \Pi_{k=m+1}^{\infty}[a_k^-, a_k^+]. \tag{4.1}$$

From the construction, it follows that $P_n(R)$ is a trapping region for the $n$th Galerkin projection of the KS equation for $n \ge m$.

From the results in [ZM], it follows that there exists $u_\nu \in R$ such that $F(u_\nu) = 0$.

2. The second part is the computation of $l$. Now, if $l < 0$, then from Theorem 3.8 it follows that $u_\nu$ is attracting.

The method of construction of the tail in self-consistent bounds, i.e., the numbers $a_k^\pm$ for $k > m$, is described in section 3.3 in [ZM], but the construction of an isolating block $N$ was not presented there; therefore, we present an outline of this algorithm here.

First, we introduce some notation and fix some terminology. We recall condition C4a from [ZM]:

C4a. Let $u \in W \oplus \Pi_{k=m+1}^{\infty}[a_k^-, a_k^+]$. Then, for $k > m$,

$$A_k u = a_k^+ \Rightarrow A_k F_\nu(u)) < 0, \tag{4.2}$$

$$A_k u = a_k^- \Rightarrow A_k F_\nu(u)) > 0. \tag{4.3}$$

Definition 4.2. *In the context of a block-decomposition of a finite-dimensional space,* $H = \bigoplus H_{(i)}$, *we consider a system of differential inclusions, which is a product of inclusions for each block of the form*

$$x_k' \in \lambda_k x_k + (b_k, B_k) \tag{4.4}$$

*or a two-dimensional block* $(k) = (k_1, k_2)$,

$$\begin{aligned} x_{k_1}' &\in \alpha_{(k)} x_{k_1} - \beta_{(k)} x_{k_2} + (b_{k_1}, B_{k_1}), \\ x_{k_1}' &\in -\beta_{(k)} x_{k_1} + \alpha_{(k)} x_{k_2} + (b_{k_2}, B_{k_2}). \end{aligned} \tag{4.5}$$

*Let* $N = \bigoplus N_{(i)}$, *where* $N_{(i)} \subset H_{(i)}$ *and*

$$\begin{aligned} N_{(i)} &= [n_{(i)}^-, n_{(i)}^+] \qquad \textit{if } \dim(i) = 1, \\ N_{(i)} &= \overline{B}(0, n_{(i)}) \qquad \textit{if } \dim(i) = 2. \end{aligned}$$

*We will say that we have* an isolation for the $(k)$-block on $N$ *if the following hold:*

$$\lambda_k n_k^+ + (b_k, B_k) < 0, \qquad \lambda_k n_k^- + (b_k, B_k) > 0 \qquad \textit{if } \dim(k) = 1. \tag{4.6}$$

*If* $\dim(k) = 2$*, we require that*

$$\lambda_{(k)} n_{(k)} + \sqrt{(b_{k_1}, B_{k_1})^2 + (b_{k_2}, B_{k_2})^2} < 0. \tag{4.7}$$

The following easy lemma explains why we care for an isolation.

Lemma 4.3. *Let a block decomposition of $H$, set $N$, and differential inclusions be as in Definition 4.2. If we have an isolation for all blocks on $N$, then the set $N$ is a trapping region. (In particular, it is an isolating block.)*

We perform our computations in an interval arithmetic [Mo]. We use the following notation and conventions.

By arabic letters we denote both single-valued objects like vectors, real numbers, and matrices and sets of these objects. Sometimes we will use square brackets, for example, $[r]$, to denote sets. Usually this will be some set constructed in an algorithm. In situations when we want to stress that we have a set in a formula involving both single-valued objects and sets together, we would rather use square brackets; hence we prefer to write $[S]$ instead of $S$ to represent a set. From this point of view, $[S]$ and $S$ are different symbols in the alphabet used to name variables, and, formally, there is no relation between the set represented by $[S]$ and the object represented by $S$. Sometimes both variables $[S]$ and $S$ are used simultaneously; usually $S \in [S]$ in this situation, but this is always stated explicitly.

For a set $[S]$ by $[S]_I$, we denote an interval hull of $[S]$, i.e., the smallest product of intervals containing $[S]$. The symbol $\mathrm{hull}(x_1, \ldots, x_k)$ will denote an interval hull of intervals $x_1, \ldots, x_k$. For any interval set $[S] = [S]_I$, by $\mathrm{m}([S])$ we will denote a center point of $[S]_I$. For any interval $[a, b]$, we define a diameter by $\mathrm{diam}([a, b]) = b - a$. For an interval vector or an interval matrix $[S] = [S]_I$, by $\mathrm{diam}([S])$ we will denote the maximum of diameters of its components. For an interval $[x^-, x^+]$, we set $right([x^-, x^+]) = x^+$ and $left([x^-, x^+]) = x^-$.

### 4.1. A detailed outline of an algorithm.

Input data:

- $m$, $M$ are dimensions describing self-consistent bounds.
- $[\nu] = \nu_0 + [-\delta\nu, \delta\nu]$ is a range of parameters.
- $x_0 \in \mathbb{R}^m$ such that $P_m F_{\nu_0}(x_0) \approx 0$; this is our candidate for a fixed point.
- $\Delta$ is a parameter defining an initial size of $N$.

1. *An approximate diagonalization of $dP_m F_{\nu_0}(x_0)$, a generation of new coordinates in $X_m = \mathbb{R}^m$, and a block decomposition of $H$.* From an approximate diagonalization of $dP_m F_{\nu_0}(x_0)$, we obtain new coordinates, which will be called *the block coordinates*. The coordinates $a_k$ will be referred to as *the standard coordinates*. The block coordinates are obtained from standard coordinates through an affine transformation $T : \mathbb{R}^m \to \mathbb{R}^m$,

$$T(x) = T_l(x - x_0), \tag{4.8}$$

where $T_l \in \mathbb{R}^{m \times m}$.

We define a block decomposition of $H = \bigoplus_{(i)} H_i$ such that, for $(i) > m$, all blocks are given by $H_{(i)} = \langle e_i \rangle$; for $(i) < m$, each block $H_{(i)}$ is an eigenspace of $dP_m F_{\nu_0}(x_0)$. Complex eigenvalues give rise to two-dimensional blocks and real eigenvalues to one-dimensional blocks. Eigenvectors from one-dimensional blocks are normalized to a unit length; in a two-dimensional block, the length of a longer vector from the pair is normalized to one.

From now on, we will change the norm in $H$ so that the blocks $H_{(i)}$ become orthogonal and, for two-dimensional blocks, the real and the imaginary parts of a complex eigenvector are orthogonal.

2. Preparation for the main loop:
   - *An initialization of variables $Iso[k]$, $1 \leq k \leq m$.* We set $Iso[k] = 0$. It will later become true (i.e., equal to 1) if we will have an isolation for the block containing a $k$th variable).
   - *An initialization of our initial guess for $N = \bigoplus_{(i) \leq m} N_{(i)}$*, where $N_{(i)} \subset H_{(i)}$. We set

$$N_{(i)} = [n^-_{(i)}, n^+_{(i)}] = [-\Delta, \Delta] \qquad \text{if } \dim(i) = 1,$$
$$N_{(i)} = \overline{B}(0, n_{(i)}) = \overline{B}(0, \Delta) \qquad \text{if } \dim(i) = 2.$$

3. Main loop:
   - *An initialization of a local variable iso_change* $= 0$. The variable *iso_change* tells us if there is any new isolation for $1 \leq k \leq m$ or if our set $N$ has changed, giving us the chance that repeating a loop once again will result in a better tail, which may produce new isolations.
   - *A computation of $W$.* $W = [T^{-1}(N)]_I$. It is enough to define $W$ as $W = T^{-1}(N)$, i.e., the set $N$ in standard coordinates. However, if we evaluate this formula in interval arithmetics, we obtain the set $[T^{-1}N]_I$, which is larger than $T^{-1}(N)$ due to round-off errors and the wrapping effect [Mo].
   - *A generation of self-consistent tail.* Using formulas derived in section 3 in [ZM], for the current values of $W$, $m$, $M$, we find $\{a^-_k, a^+_k\}$ such that conditions C1, C2, C3, and C4a are satisfied on $W \oplus \Pi^{\infty}_{k=m+1}[a^-_k, a^+_k]$.
   In principle, the procedure of generation $\{a^{\pm}_k\}$ may fail; in this case, we interrupt the algorithm and return *fail*.
   - *A computation of an influence of the tail $V = \Pi^{\infty}_{k=m+1}[a^-_k, a^+_k]$ onto the $m$-dimensional Galerkin projection.* Using formulas from section 3 in [ZM], we find an interval vector $[\epsilon] \subset \mathbb{R}^n$ such that

$$P_m(F_\nu(x)) - P_m(F_\nu(P_m x)) \subset [\epsilon] \qquad \text{for } x \in W \oplus V. \tag{4.9}$$

   Our goal for the next step is to construct an isolating block $N$ for an equation (in fact a differential inclusion)

$$x' \in P_m(F_\nu(x)) + [\epsilon], \qquad \text{where } x \in W. \tag{4.10}$$

   - *A transformation of (4.10) to the block coordinates and "an interval diagonalization."* We transform (4.10) into the block coordinates; as a result, we obtain for $\nu \in \nu_0 + [-\delta\nu, \delta\nu]$, for one-dimensional blocks $(i)$,

$$x'_i \in \lambda_i(\nu) x_i + f_i(x) + [\tilde{\epsilon}_i] \tag{4.11}$$

   and, for two-dimensional blocks $(i) = (i_1, i_2)$,

$$x'_{i_1} \in \alpha_{(i)}(\nu) x_{i_1} + \beta_{(i)}(\nu) x_{i_2} + f_{i_1}(x) + [\tilde{\epsilon}_{i_1}],$$
$$x'_{i_2} \in -\beta_{(i)}(\nu) x_{i_1} + \alpha_{(i)}(\nu) x_{i_2} + f_{i_2}(x) + [\tilde{\epsilon}_{i_2}]. \tag{4.12}$$

Observe that, since we are using the block coordinates, which diagonalize $dP_mFP_m$, for a small $N$ the values $f_i(x)$ for $x \in N$ will usually be very small.
For $1 \leq i \leq m$, we compute an interval $(b_i, B_i)$ such that

$$f_i(W) + \tilde{\epsilon}_i \subset (b_i, B_i). \tag{4.13}$$

Instead of (4.11) and (4.12), we will now consider the equations

$$x'_i \in \lambda_i(\nu)x_i + (b_i, B_i) \tag{4.14}$$

and

$$\begin{aligned} x'_{i_1} &\in \alpha_{(i)}x_{i_1} + \beta_{(i)}x_{i_2} + (b_{i_1}, B_{i_1}), \\ x'_{i_2} &\in -\beta_{(i)}x_{i_1} + \alpha_{(i)}x_{i_2} + (b_{i_2}, B_{i_2}), \end{aligned} \tag{4.15}$$

respectively.
Let us stress that, in our computations, we have uniform bounds for $\lambda_{(i)}(\nu)$, $\alpha_{(i)}(\nu)$, and $\beta_{(i)}(\nu)$. Namely, we have intervals $[\lambda_{(i)}]$, $[\alpha_{(i)}]$, and $[\beta_{(i)}]$ such that, for all $\nu \in [\nu_0 - \delta\nu, \nu_0 + \delta\nu]$, the following hold:

$$\lambda_{(i)}(\nu) \in [\lambda_{(i)}], \quad \alpha_{(i)}(\nu) \in [\alpha_{(i)}], \quad \beta_{(i)}(\nu) \in [\beta_{(i)}].$$

- *An isolation for* $1 \leq i \leq m$. First observe that, since we are looking for an attracting fixed point, then $\lambda_i < 0$ and $\alpha_{(i)} < 0$ (provided $x_0$ is a good approximation).
  For each block, we try to find an isolation as follows: for one-dimensional block $(i)$, we set

$$d_i^+ = \text{right}\left(-\frac{B_i}{[\lambda_i]}\right), \qquad d_i^- = \text{left}\left(-\frac{b_i}{[\lambda_i]}\right). \tag{4.16}$$

  Now if

$$[d_i^-, d_i^+] \subset [n_i^-, n_i^+], \tag{4.17}$$

  then an easy computation shows that we have an isolation for the $(i)$th block.
  For two-dimensional blocks $(i) = (i_1, i_2)$, we set

$$d_{(i)} = \text{right}\left(-\frac{B}{[\alpha_{(i)}]}\right), \tag{4.18}$$

  where $B = \text{right}(\sqrt{(b_{i_1}, B_{i_1})^2 + (b_{i_2}, B_{i_2})^2})$. It is easy to see that we have an isolation for the $(i)$th block on $N$ if

$$n_{(i)} \geq d_{(i)}. \tag{4.19}$$

  If (4.17) holds for one-dimensional block $(i)$, then we set

$$\begin{aligned} iso\_change &= 1, \\ n_i^+ = d_i^+, &\quad n_i^- = d_i^-, \\ Iso[i] &= 1. \end{aligned}$$

If (4.19) holds for two-dimensional block $(i) = (i_1, i_2)$, then we set

$$iso\_change = 1,$$
$$n_{(i)} = d_{(i)},$$
$$Iso[i_1] = 1, \qquad Iso[i_2] = 1.$$

Observe that, with these updates, we achieve the following: if $iso\_change = 1$, then the current set $N$ is a proper subset of the set $N$ at the beginning of the loop. This guarantees that $W$ and then also $[\epsilon]$ will be smaller in the next iterate. This implies also that, if in one loop we have an isolation for a block $(i)$, then we will have an isolation for this block for all following iterations of the loop, and it creates a possibility for obtaining an isolation for other blocks in the next iterations.

- *A verification of an isolation for all blocks.* We check if, for all $1 \leq k \leq m$, $Iso[k] = 1$. If this is the case, then we leave the loop because $N$ is the desired trapping region. Otherwise, we check if $iso\_change = 0$. If this is the case, then the algorithm failed. If $iso\_change = 1$, we repeat the loop again.

4. Computation of $l$: From previous steps, we have a block decomposition of $H$, a change of coordinates $T$, a trapping region $N$, and a tail $[a_k^-, a_k^+]_{k>m}$.

We compute $l$ using formulas from section 5 on the set $W = [T^{-1}(N)]_I \oplus \Pi_{k>m}[a_k^-, a_k^+]$.

5. Output: We set $x_c = \mathrm{m}(N)$ (a center point of $N$ ). Let $x_* = T^{-1}x_c$. $x_*$ is a center of a trapping region $N$ expressed in the standard coordinates. We estimate $|y - x_*|$ in various norms for all $y \in T^{-1}(N) \oplus \Pi_{k>m}[a_k^-, a_k^+]$ (for example, $L_2$, $H^1$ etc.). If $l < 0$, then we can conclude that, close to $x_*$, there exists an attracting fixed point; otherwise, we can claim only the existence of a fixed point.

End of an algorithm.

**4.2. Numerical data from the proof of Theorem 4.1.** We have chosen $m = 3$ and $M = 10$. In fact, to complete the full algorithm successfully without checking that $l < 0$, i.e., to prove just the existence of a fixed point, it is enough to take $m = 2$ (see the proof of Theorem 4.1 in [ZM]), but, in this case, we were unable to verify that $l < 0$.

Other starting parameters for the algorithm were given by

$$x_0 = (0.712361, \ -0.12324, \ 0.0101787),$$
$$\Delta = 0.03125.$$

$x_0$ was found by simply integrating forward a three-dimensional Galerkin projection of (1.1). We tried first $\Delta = 10^{-5}$ and then $\Delta = 5\Delta$ (we multiplied the current value of $\Delta$ by 5) until we were able to successfully complete the algorithm.

From the approximate diagonalization, we list approximate eigenvalues and several most significant digits of interval matrixes $T_l$ and $T_l^{-1}$. Diameters of entries in $T_l$ and $T_l^{-1}$ were smaller than $10^{-15}$.

$$\lambda_1 \approx -51.46617, \qquad \lambda_2 \approx -7.7545, \qquad \lambda_3 \approx -0.52575.$$

We see that, in our block decomposition, we have only one-dimensional blocks.

$$T_l = \begin{bmatrix} -0.0090621 & 0.09949 & 1.0087 \\ -0.39312 & -1.1041 & -0.069407 \\ -1.1408 & -0.21674 & -0.0066056 \end{bmatrix},$$

$$T_l^{-1} = \begin{bmatrix} 0.0065846 & 0.18519 & -0.94042 \\ -0.065071 & -0.97779 & 0.33745 \\ 0.99786 & 0.098106 & -0.041732 \end{bmatrix}.$$

We obtained an isolation for $1 \le i \le 3$ after four iterates of the main loop for the set $N$ are given by

$$N = [-3.176878e-04, 2.272739e-04] \times [-3.618637e-03, 3.756375e-03] \\ \times [-2.566221e-02, 2.711549e-02].$$

For $l$, we have

$$l < -0.05716.$$

Below we list some other data obtained in the algorithm. The set $W = [T^{-1}(N)]_I$ is given by

$$\begin{aligned} W &= [a_1^-, a_1^+] \times [a_2^-, a_2^+] \times [a_3^-, a_3^+], \\ [a_1^-, a_1^+] &= 0.711691 + [-0.0255012, 0.0255012], \\ [a_2^-, a_2^+] &= -0.123059 + [-0.0125283, 0.0125283], \\ [a_3^-, a_3^+] &= 0.01011 + [-0.00173494, 0.00173494]. \end{aligned}$$

In Table 4.1, we list $a_k^\pm$ for $k > 4$.

**Table 4.1**

*Estimates for the intervals $[a_k^-, a_k^+]$ representing self-consistent a priori bounds in the proof of Theorem 4.1.*

| $k$ | $[a_k^-, a_k^+]$ |
|---|---|
| 4 | $-6.77647 \cdot 10^{-4} + [-1.48185, 1.48185] \cdot 10^{-4}$ |
| 5 | $3.95994 \cdot 10^{-5} + [-1.10021, 1.10021] \cdot 10^{-5}$ |
| 6 | $-2.14113 \cdot 10^{-6} + [-7.10907, 7.10907] \cdot 10^{-7}$ |
| 7 | $1.09725 \cdot 10^{-7} + [-4.21541, 4.21541] \cdot 10^{-8}$ |
| 8 | $-5.41181 \cdot 10^{-9} + [-2.35893, 2.35893] \cdot 10^{-9}$ |
| 9 | $2.60507 \cdot 10^{-10} + [-2.10033, 2.10033] \cdot 10^{-10}$ |
| 10 | $-1.22454 \cdot 10^{-11} + [-3.57734, 3.57734] \cdot 10^{-10}$ |
| $> 10$ | $[-1, 1] \cdot 4176.07/k^{10}$ |

For $[\epsilon] = ([\epsilon_1], [\epsilon_2], [\epsilon_3])$, we obtained the following numbers:

$$\begin{aligned} [\epsilon_1] &= -1.42733 \cdot 10^{-5} + [-5.37438, 5.37438] \cdot 10^{-6}, \\ [\epsilon_2] &= 0.000342671 + [-0.000107623, 0.000107623], \\ [\epsilon_3] &= -0.00294653 + [-0.00074762, 0.00074762]. \end{aligned}$$

After passing to the block coordinates and an interval diagonalization, we obtained on $N \oplus \Pi_{k>m}[a_k^-, a_k^+]$

$$x_1' \in [-0.01608764, 0.01150572] + [-52.28307, -50.65041]x_1,$$
$$x_2' \in [-0.02740801, 0.02844858] + [-7.932687, -7.574724]x_2,$$
$$x_3' \in [-0.01291743, 0.01364734] + [-0.5486830, -0.5033749]x_3.$$

In Table 4.2, we list the computed upper bounds for $l_i$ on $N \oplus \Pi_{k>m}[a_k^-, a_k^+]$. It is easy to see that $l = l_3 < -0.05716$.

**Table 4.2**

*Estimates from above on $l_i$ from the computation of $l$ in the proof of Theorem 4.1.*

| $k$ | $l_k$ |
|---|---|
| 1 | -44.76 |
| 2 | -6.06 |
| 3 | -0.05716 |
| 4 | -160.6 |
| 5 | -425.8 |
| 6 | -914.7 |
| 7 | -1726.9 |
| 8 | -2979.6 |
| 9 | -4807.8 |
| 10 | -7364.5 |
| $> 10$ | -10820.7 |

Observe that the value of $l = -0.05716$ is close to zero. This is essentially due to the fact that we wanted to extend as much as possible the parameter interval. If we just choose $\nu = 0.75$ with the same $x_0$, $m$, and $M$, then we obtain $l = -0.348938$. By increasing $m$ and $M$, we can further decrease $l$ up to $\lambda_3$. For example, for $m = 15$ and $M = 45$, we obtained $l = -0.52581$.

**5. Details for the KS equation.** The goal of this section is to derive formulas from which, for a given block decomposition and trapping region, the constants $l_{(i)}$ for the KS equation may be computed, thus implementing step 4 of the algorithm of section 4.1.

**5.1. The KS equation in Fourier representation.** For the KS equation in one space dimension with periodic and odd boundary conditions, we have the following infinite ladder of equations for the Fourier coefficients (see [ZM]):

$$\dot{a}_k = F_k(a) = \lambda_k a_k + N_k(a), \tag{5.1}$$
$$\lambda_k = k^2(1 - \nu k^2), \tag{5.2}$$
$$N_k(a) = -k \sum_{n=1}^{k-1} a_n a_{k-n} + 2k \sum_{n=1}^{\infty} a_n a_{n+k}. \tag{5.3}$$

Hence we obtain

$$\frac{\partial N_i}{\partial a_j} = 2ia_{i+j} \quad \text{for } i = j,$$

$$\frac{\partial N_i}{\partial a_j} = -2ia_{i-j} + 2ia_{i+j} \quad \text{for } j < i,$$
$$\frac{\partial N_i}{\partial a_j} = 2ia_{j-i} + 2ia_{i+j} \quad \text{for } j > i,$$
$$\frac{\partial F_i}{\partial a_j} = i^2(1-\nu i^2)\delta_{ij} + \frac{\partial N_i}{\partial a_j}.$$

**5.2. Coordinate change and block-decomposition.** The following lemma does not require any proof.

Lemma 5.1. *Let $A : H \to H$ be a linear coordinate change of the form*

$$A : X_m \oplus Y_m \to X_m \oplus Y_m,$$
$$A(x \oplus y) = Ax \oplus y.$$

*Let $\tilde{F} = A \circ F \circ A^{-1}$. ($\tilde{F}$ is $F$ expressed in new coordinates.)*

$$\frac{\partial \tilde{F}_i}{\partial x_j} = \sum_{k,l=1}^{m} A_{ik} \frac{\partial F_k}{\partial x_l} A^{-1}_{lj} \quad \text{for } i \le m \text{ and } j \le m,$$
$$\frac{\partial \tilde{F}_i}{\partial x_j} = \sum_{k\le m} A_{ik} \frac{\partial F_k}{\partial x_j} \quad \text{for } i \le m \text{ and } j > m,$$
$$\frac{\partial \tilde{F}_i}{\partial x_j} = \sum_{l\le m} \frac{\partial F_i}{\partial x_l} A^{-1}_{lj} \quad \text{for } i > m \text{ and } j \le m,$$
$$\frac{\partial \tilde{F}_i}{\partial x_j} = \frac{\partial F_i}{\partial x_j} \quad \text{for } i > m \text{ and } j > m.$$

Consider now the KS equation and assume that $W = N \oplus \Pi_{k=m+1}^{\infty}[a_k^-, a_k^+]$ is a trapping region representing self-consistent bounds for a fixed point. Let the numbers $m < M$ be as in conditions C1, C2, and C3, and we assume that $a_k^{\pm} = \pm\frac{C}{k^s}$ for $k > M$ (as in [ZM]).

Let $A \in \mathbb{R}^{m\times m}$ be a coordinate change around an approximate fixed point in $X_m$ for the $m$-dimensional Galerkin projection of (5.1). This matrix induces a coordinate change in $H$. For our purpose, it is optimal to choose $A$ so that the $m$-dimensional Galerkin projection of $F$ is very close to a diagonal matrix (or to a block diagonal matrix when complex eigenvalues are present).

From now on, we will use these new coordinates on $H$. We also change the norm so that the new coordinates become orthogonal. We define the splitting of $P_m H$ into blocks which are either two-dimensional (complex eigenvalue) or one-dimensional (real eigenvalue). For the KS equation, there was no need to consider more complicated situations. For $(i) > m$, all blocks are one-dimensional. (These coordinates are not affected by our coordinate change.)

We would like to derive the formula for

$$l_{(i)} := \sup_{x\in W} \mu\left(\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}}(x)\right) + \sum_{(j),(j)\neq(i)} \sup_{x\in W} \left|\frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}}(x)\right|. \tag{5.4}$$

We define $S(l)$ by

$$S(l) = \sum_{k \geq l} \sup_{a \in W} |a_k|.$$

We estimate $S(l)$ from above using the following lemma.

Lemma 5.2. *Assume that* $|a_k(W)| \leq \frac{C}{k^s}$ *for* $k > M$, $s > 1$; *then*

$$S(l) < \sum_{k=l}^{M} |a_k(W)| + \frac{C}{(s-1)M^{s-1}} \quad \text{for } l \leq M,$$

$$S(l) < \frac{C}{(s-1)(l-1)^{s-1}} \quad \text{for } l > M.$$

*Proof.* Observe that

$$\sum_{k=l}^{\infty} \frac{1}{k^s} < \int_{l-1}^{\infty} \frac{dx}{x^s} = \frac{1}{(s-1)(l-1)^{s-1}}. \tag{5.5}$$ ■

We set

$$\overline{S}(l) = \sum_{k=l}^{M} |a_k(W)| + \frac{C}{(s-1)M^{s-1}} \quad \text{for } l \leq M, \tag{5.6}$$

$$\overline{S}(l) = \frac{C}{(s-1)(l-1)^{s-1}} \quad \text{for } l > M. \tag{5.7}$$

**5.3. Formulas for one-dimensional blocks.** Observe that, if $\dim(i) = 1$, then $\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}} \in \mathbb{R}$; hence

$$\mu\left(\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}}\right) = \frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}}. \tag{5.8}$$

Lemma 5.3. *Assume* $(i) \leq m$ *and* $\dim\ (i) = 1$.

$$\begin{aligned} &\sup_{x \in W} \mu\left(\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}}(x)\right) + \sum_{(j) \neq (i)} \sup_{x \in W} \left|\frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}}(x)\right| \\ &\leq \bar{l}_i := \sup_{x \in W} \frac{\partial \tilde{F}_i}{\partial x_i}(x) + \sum_{j \neq i, j \leq m} \sup_{x \in W} \left|\frac{\partial \tilde{F}_i}{\partial x_j}(x)\right| \\ &+ 2 \sum_{k \leq m} k|A_{ik}|(\overline{S}(m-k+1) + \overline{S}(m+k+1)). \end{aligned}$$

*Proof.* Observe that, when $(j) = (j_1, j_2)$ is a two-dimensional block, then we need to compute the norm of $[\frac{\partial \tilde{F}_i}{\partial x_{j_1}}(x), \frac{\partial \tilde{F}_i}{\partial x_{j_2}}(x)]$. It is easy to see that this norm is less than or equal to

$$\left|\frac{\partial \tilde{F}_i}{\partial x_{j_1}}(x)\right| + \left|\frac{\partial \tilde{F}_i}{\partial x_{j_2}}(x)\right|.$$

This means that ignoring the block structure is safe (we have an inequality in the correct direction); hence we obtain

$$\sum_{(j)\neq(i)} \left|\frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}}(W)\right| \leq \sum_{j\neq i} \left|\frac{\partial \tilde{F}_i}{\partial x_j}(W)\right|. \tag{5.9}$$

To finish the proof, it is enough to show that

$$\sum_{j>m} \left|\frac{\partial \tilde{F}_i}{\partial x_j}(W)\right| \leq 2\sum_{k\leq m} k|A_{ik}|(S(m-k+1)+S(m+k+1)). \tag{5.10}$$

Observe that, from Lemma 5.1, it follows that

$$\begin{aligned}\sum_{j>m} \left|\frac{\partial \tilde{F}_i}{\partial x_j}(W)\right| &\leq \sum_{j>m}\sum_{k\leq m} |A_{ik}|2k(|a_{j-k}(W)|+|a_{j+k}(W)|) \\ &= \sum_{k\leq m} |A_{ik}|2k\left(\sum_{j>m} |a_{j-k}(W)|+|a_{j+k}(W)|\right) \\ &= \sum_{k\leq m} |A_{ik}|2k\left(S(m-k+1)+S(m+k+1)\right). \qquad \blacksquare\end{aligned}$$

Observe that, from our assumptions about the decomposition of $H$, it follows that all blocks $(i)$, such that $(i)>m$, are one-dimensional.

Lemma 5.4. *For $m<i\leq M$, we have*

$$\begin{aligned}&\sup_{x\in W} \mu\left(\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}}(x)\right) + \sum_{(j),(j)\neq(i)} \sup_{x\in W}\left|\frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}}(x)\right| \\ &\leq \bar{l}_i = i^2(1-\nu i^2) + \sum_{j\leq M}\sup_{x\in W}\left|\frac{\partial \tilde{N}_i}{\partial x_j}(x)\right| + 2i(\overline{S}(M+1-i)+\overline{S}(i+M+1)).\end{aligned}$$

*Proof.* Just as in the proof of Lemma 5.3, we can ignore the block structure here. It is easy to see that

$$\sup_{x\in W} \frac{\partial \tilde{F}_i}{\partial x_i}(x) + \sum_{j\neq i}\sup_{x\in W}\left|\frac{\partial \tilde{F}_i}{\partial x_j}(x)\right| \leq i^2(1-\nu i^2) + \sum_{j=1}^{\infty}\sup_{x\in W}\left|\frac{\partial \tilde{N}_i}{\partial x_j}(x)\right|. \tag{5.11}$$

Therefore, to finish the proof, it is enough to show that

$$\sum_{j=M+1}^{\infty}\sup_{x\in W}\left|\frac{\partial \tilde{N}_i}{\partial x_j}(x)\right| < 2i(S(M+1-i)+S(i+M+1)). \tag{5.12}$$

We proceed as follows:

$$\begin{aligned}&\sum_{j=M+1}^{\infty}\sup_{x\in W}\left|\frac{\partial \tilde{N}_i}{\partial x_j}(x)\right| = \sum_{j=M+1}^{\infty}\sup_{x\in W}\left|\frac{\partial N_i}{\partial x_j}(x)\right| \\ &\leq \sum_{j=M+1}^{\infty} 2i(|a_{j-i}(W)|+|a_{i+j}(W)|) \leq 2i(S(M+1-i)+S(M+1+i)). \qquad \blacksquare\end{aligned}$$

Lemma 5.5. *For* $(i) > m$, *we have*

$$\sup_{x\in W} \mu\left(\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}}(x)\right) + \sum_{(j)\neq(i)} \sup_{x\in W}\left|\frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}}(x)\right| \leq \bar{l}_i := i^2(1-\nu i^2)$$
$$+ \ 2im(\overline{S}(i-m)+\overline{S}(i+1)) \max_{k,l=1,\dots,m} |A^{-1}_{kl}|$$
$$+ \ 2i(\overline{S}(i+m+1)+2\overline{S}(1)).$$

*Proof.* Just as in the proof of Lemma 5.3, we can ignore the block structure here. It is easy to see that

$$\sup_{x\in W} \frac{\partial \tilde{F}_i}{\partial x_i}(x) + \sum_{j\neq i} \sup_{x\in W}\left|\frac{\partial \tilde{F}_i}{\partial x_j}(x)\right| \leq i^2(1-\nu i^2) + \sum_{j=1}^{\infty} \sup_{x\in W}\left|\frac{\partial \tilde{N}_i}{\partial x_j}(x)\right|. \tag{5.13}$$

Therefore, to finish the proof, it is enough to show that

$$\sum_{j=1}^{m} \sup_{x\in W}\left|\frac{\partial \tilde{N}_i}{\partial x_j}(x)\right| \leq 2im(S(i-m)+S(i+1)) \max_{k,l=1,\dots,m} |A^{-1}_{kl}|, \tag{5.14}$$

$$\sum_{j=m+1}^{\infty} \sup_{x\in W}\left|\frac{\partial \tilde{N}_i}{\partial x_j}(x)\right| \leq 2i(S(i+m+1)+2S(1)). \tag{5.15}$$

To prove (5.14), observe that

$$\begin{aligned}
\sum_{j=1}^{m} \sup_{x\in W}\left|\frac{\partial \tilde{N}_i}{\partial x_j}(x)\right| &= \sum_{j=1}^{m} \sup_{x\in W}\left|\sum_{l=1}^{m} \frac{\partial N_i}{\partial x_l}(x) A^{-1}_{lj}\right| \\
&\leq \sum_{j=1}^{m}\sum_{l=1}^{m} 2i(|a_{i-l}(W)|+|a_{i+l}(W)|)|A^{-1}_{lj}| \\
&\leq 2i\sum_{j=1}^{m}(S(i-m)+S(i+1)) \max_{k,l=1,\dots,m}|A^{-1}_{kl}| \\
&= 2im(S(i-m)+S(i+1)) \max_{k,l=1,\dots,m}|A^{-1}_{kl}|.
\end{aligned}$$

To prove (5.15), we proceed as follows:

$$\begin{aligned}
\sum_{j=m+1}^{\infty} \sup_{x\in W}\left|\frac{\partial \tilde{N}_i}{\partial x_j}(x)\right| &= \sum_{j=m+1}^{\infty} \sup_{x\in W}\left|\frac{\partial N_i}{\partial x_j}(x)\right| \\
&\leq \sum_{m<j<i} (2i(|a_{i-j}(W)|+|a_{i+j}(W)|)) \\
&\quad + 2i|a_{2i}(W)| + \sum_{j>i} 2i(|a_{j-i}(W)|+|a_{i+j}(W)|) \\
&\leq 2i\left(\sum_{j>m}|a_{i+j}(W)| + \sum_{m<j<i}|a_{i-j}(W)| + \sum_{j>i}|a_{j-i}(W)|\right) \\
&< 2i\left(S(i+m+1)+2S(1)\right). \qquad \blacksquare
\end{aligned}$$

The following lemma shows how to handle the case of large $i$.

Lemma 5.6. *If, for some $n > m$, $\bar{l}_n < 0$, then*

$$0 > \bar{l}_i > \bar{l}_j \qquad \text{for} \qquad i < j, \quad i \geq n. \tag{5.16}$$

*Proof.* From Lemma 5.5, it follows that

$$\bar{l}_i = i((i - \nu i^3) + 2m(\overline{S}(i-m) + \overline{S}(i+1))a + 2(\overline{S}(i+m+1) + 2\overline{S}(1))),$$

where $a = \max_{k,l=1,\dots,m} |A^{-1}_{kl}|$.

Hence

$$\bar{l}_i = i((i - \nu i^3) + f(i)), \tag{5.17}$$

where $f(i)$ is a positive decreasing function of $i$. Since $\bar{l}_n < 0$, then $(n - \nu n^3) < 0$ also, and it is easy to see that the function $i \mapsto (i - \nu i^3)$ is decreasing and negative for $i \geq n$. ∎

**5.4. Formulas for "complex" blocks.** The purpose of this subsection is to derive a formula for $l_{(i)}$ in case of a two-dimensional block from the diagonalization corresponding to a complex eigenvalue. The main results are summarized in Lemmas 5.8 and 5.9.

Lemma 5.7. *Let $Q \in \mathbb{R}^{2\times 2}$; then (in the Euclidean norm)*

$$|Q| \leq \sqrt{Q_{11}^2 + Q_{12}^2} + \sqrt{Q_{21}^2 + Q_{22}^2}.$$

*Proof.* Let $v = (v_1, v_2)$; then

$$|Qv| \leq |Q_{11}v_1 + Q_{12}v_2| + |Q_{21}v_1 + Q_{22}v_2| \leq \sqrt{Q_{11}^2 + Q_{12}^2}|v| + \sqrt{Q_{21}^2 + Q_{22}^2}|v|. \qquad ∎$$

Lemma 5.8. *If $(i) = (i_1, i_2)$, then*

$$\sup_{x\in W} \mu\left(\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}}\right) \leq \max_{k=1,2}\left(\sup_{x\in W} \frac{\partial \tilde{F}_{i_k}}{\partial x_{i_k}}(x)\right) + \sup_{x\in W} 1/2 \left|\frac{\partial \tilde{F}_{i_1}}{\partial x_{i_2}}(x) + \frac{\partial \tilde{F}_{i_2}}{\partial x_{i_1}}(x)\right|.$$

*Proof.* The proof is an immediate consequence of Theorem 3.5 and the Gershgorin theorem (see [QSS, Property 5.2]). ∎

Observe that, from the diagonalization of a block corresponding to a complex eigenvalue, we obtain

$$\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}} \approx \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix};$$

hence $\sup_{x\in W} 1/2|\frac{\partial \tilde{F}_{i_1}}{\partial x_{i_2}}(x) + \frac{\partial \tilde{F}_{i_2}}{\partial x_{i_1}}(x)|$ is usually very small.

The following lemma takes care of nondiagonal terms.

Lemma 5.9. *If $(i) = (i_1, i_2)$, $(i) \leq m$, then*

$$\sum_{(j),(j)\neq(i)} \sup_{x\in W} \left|\frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}}(x)\right| \leq \sum_{j\leq M, j\neq i_1,i_2} \sup_{x\in W} \sqrt{\left(\frac{\partial \tilde{F}_{i_1}}{\partial x_j}\right)^2 + \left(\frac{\partial \tilde{F}_{i_2}}{\partial x_j}\right)^2}$$
$$+ \sum_{l=1,2} \sum_{k\leq m} 2|A_{i_l,k}|k(\overline{S}(M+1-k) + \overline{S}(M+1+k)).$$

*Proof.* If $\dim(j) = 1$, then

$$\frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}}(x) = \left[\frac{\partial \tilde{F}_{i_1}}{\partial x_j}(x), \frac{\partial \tilde{F}_{i_2}}{\partial x_j}(x)\right]. \tag{5.18}$$

Therefore, we obtain

$$\left|\frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}}(x)\right| = \sqrt{\left(\frac{\partial \tilde{F}_{i_1}}{\partial x_j}(x)\right)^2 + \left(\frac{\partial \tilde{F}_{i_2}}{\partial x_j}(x)\right)^2}. \tag{5.19}$$

From Lemma 5.7, it follows that we can ignore the block structure for all blocks different from $(i)$ and use the above formula for all coordinates.

Therefore, we have

$$\sum_{(j)\neq(i)} \sup_{x\in W} \left|\frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}}(x)\right| \leq \sum_{j\neq i_1,i_2} \sup_{x\in W} \sqrt{\left(\frac{\partial \tilde{F}_{i_1}}{\partial x_j}\right)^2 + \left(\frac{\partial \tilde{F}_{i_2}}{\partial x_j}\right)^2}. \tag{5.20}$$

To finish the proof, it is enough to show that

$$\sum_{j>M} \sup_{x\in W} \sqrt{\left(\frac{\partial \tilde{F}_{i_1}}{\partial x_j}\right)^2 + \left(\frac{\partial \tilde{F}_{i_2}}{\partial x_j}\right)^2}$$
$$\leq \sum_{l=1,2} \sum_{k\leq m} 2|A_{i_l,k}| k (S(M+1-k) + S(M+1+k)).$$

To make the notation less cumbersome, we will drop $\sup_{x\in W}$ from the computations below:

$$\sum_{j>M} \sqrt{\left(\frac{\partial \tilde{F}_{i_1}}{\partial x_j}\right)^2 + \left(\frac{\partial \tilde{F}_{i_2}}{\partial x_j}\right)^2} \leq \sum_{j>M} \left|\frac{\partial \tilde{F}_{i_1}}{\partial x_j}\right| + \sum_{j>M} \left|\frac{\partial \tilde{F}_{i_2}}{\partial x_j}\right|. \tag{5.21}$$

We have, for $l = 1, 2$ (observe that $i_l \leq m$),

$$\begin{aligned}\sum_{j>M} \left|\frac{\partial \tilde{F}_{i_l}}{\partial x_j}\right| &\leq \sum_{j>M} \sum_{k\leq m} |A_{i_l,k}| \left|\frac{\partial F_k}{\partial x_j}\right| \\ &= \sum_{k\leq m} |A_{i_l,k}| \sum_{j>M} \left|\frac{\partial F_k}{\partial x_j}\right| \leq \sum_{k\leq m} 2k|A_{i_l,k}| \sum_{j>M} (|a_{j-k}| + |a_{j+k}|) \\ &\leq \sum_{k\leq m} 2k|A_{i_l,k}| \left(S(M+1-k) + S(M+1+k)\right).\end{aligned}$$

This finishes the proof. ■

**6. Conclusions and future work.** We have shown that we can prove rigorously the existence of branches of attracting fixed points for the KS equation with odd periodic boundary conditions. Below we indicate some further possible developments and applications of the method:

- a rigorous steady state bifurcation diagram for the KS equation,
- applications of other dissipative PDEs, e.g., Navier–Stokes equations with periodic boundary conditions on the plane,
- an automatization of generation of formulas for tail (a content of section 5 in this paper and section 3 in [ZM]) for KS and other equations.

## REFERENCES


[HNW] E. Hairer, S. P. Nørsett, and G. Wanner, *Solving Ordinary Differential Equations* I*, Nonstiff Problems*, Springer-Verlag, Berlin, Heidelberg, 1987.

[JKT] M. Jolly, I. Kevrekidis, and E. Titi, *Approximate inertial manifolds for the Kuramoto–Sivashinsky equation: Analysis and computations*, Phys. D, 44 (1990), pp. 38–60.

[Mo] R. E. Moore, *Methods and Applications of Interval Analysis*, SIAM Stud. Appl. Math. 2, SIAM, Philadelphia, 1979.

[QSS] A. Quarteroni, R. Sacco, and F. Saleri, *Numerical Mathematics*, Texts Appl. Math. 37, Springer-Verlag, New York, 2000.

[Z] P. Zgliczynski, *Trapping Regions and an ODE-Type Proof of an Existence and Uniqueness for Navier-Stokes Equations with Periodic Boundary Conditions on the Plane*, available online at http://arXiv.org/abs/math/0103053; see also http://www.im.uj.edu.pl/~zgliczyn.

[ZM] P. Zgliczynski and K. Mischaikow, *Rigorous numerics for partial differential equations: The Kuramoto-Sivashinsky equation*, Found. Comput. Math., 1 (2001), pp. 255–288.